Approximate solution of variational problems for the mixed type nonlocal functionals

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1 Statement of problems

There is considered the problem of extremum of the mixed type nonlocal functional

$$J(u) = \int_{t_0}^{t_1} dt \int_{s_0}^{s_1} F[t, s, u(t, s - r_1), \ldots, u(t, s) \ldots, u(t, s + r_2), \\ u_t(t, s - r_1), \ldots, u_t(t, s), \ldots, u_t(t, s + r_2)] ds, \tag{1}$$

where $u : \mathbb{R}^2 \to \mathbb{R}^1$, $s_1 - s_0 > r_1 + r_2$,r_1,r_2 are some integers; $u_t(t, s)$ denotes the partial derivative of $u(t, s)$ with respect to t.

On the set $E_0 = \{(t, s)|t \in [t_0, t_1], s \in [s_0 - r_1, s_0 + r_2)\}$ the boundary function $\varphi(t, s)$ is given; on the set $E_1 = \{(t, s)|t \in [t_0, t_1], s \in (s_1 - r_1, s_1 + r_2]\}$ the boundary function $\psi(t, s)$ is given. On the intervals $G_0 = \{(t, s)|t = t_0, s \in (s_0 + r_2, s_1 - r_1)\}$ and $G_1 = \{(t, s)|t = t_1, s \in (s_0 + r_2, s_1 - r_1)\}$ there are given functions $\mu(s)$ and $\nu(s)$.

The problem of extremum of the functional (1) is considered with boundary conditions

$$\begin{aligned} u(t, s) &= \varphi(t, s), \quad (t, s) \in E_0; \\ u(t, s) &= \psi(t, s), \quad (t, s) \in E_1; \end{aligned} \tag{2}$$

$$\begin{aligned} u(t, s) &= \mu(s), \quad (t, s) \in G_0; \\ u(t, s) &= \nu(s), \quad (t, s) \in G_1; \end{aligned} \tag{3}$$

Define the set

$$Q = (t_0, t_1) \times (s_0 + r_2, s_1 - r_1).$$

Introduce the Hilbert space $L_2(Q)$ of measurable real functions that are square integrable on Q, with the scalar product

$$(u, v) = \int\limits_Q u(t, s)\, v(t, s)\, dt\, ds,$$

and the Sobolev space $W^{p,0}(Q)$ of functions from $L_2(Q)$ together with the generalized derivatives in t of order p inclusively, and with the scalar product

$$(u, v)_p = \int\limits_Q \sum_{i=0}^{p} u_t^{(p)}(t, s)\, v_t^{(p)}(t, s)\, dt\, ds.$$

and the norm $\| \cdot \|_p$.

Denote $\mathbb{H}(Q)$ the space of differentiable with respect to t almost everywhere on Q functions $u(t, s)$ such that $u \in W^{1,0}(Q)$.

It is assumed that admissible functions $u(t, s) \in \mathbb{H}(Q)$ and $\varphi \in \mathbb{H}(E_0)$, $\psi \in \mathbb{H}(E_1)$; the functions μ and ν are integrable and almost everywhere bounded on G_0 and G_1 correspondingly.

The problem (1),(2),(3) was studied in [1] and it was proved the following analog to the Euler necessary condition.

Define the function

$$\begin{aligned} &\Phi(t, s, u(t, s - r_1 - r_2), \dots, u(t, s), \dots, u(t, s + r_1 + r_2), \\ &\qquad u_t(t, s - r_1 - r_2), \dots, u_t(t, s), \dots, u_t(t, s + r_1 + r_2)) \qquad (4) \\ &:= \sum_{i=-r_2}^{r_1} F(t, s + i, u(t, s - r_1 + i), \dots, u(t, s + r_2 + i), \\ &\qquad\qquad u_t(t, s - r_1 + i), \dots, u_t(t, s + r_2 + i)). \end{aligned}$$

Theorem 1 *Suppose that functional* (1) *attains an extremum on* $u(t, s)$ *in* $\mathbb{H}$. *Then* $u(t, s)$ *satisfies almost everywhere on* Q *the equation*

$$\Phi_{u(t,s)} - \frac{d}{dt}\Phi_{u_t(t,s)} = 0. \qquad (5)$$

Equation (5) is a generalized Euler's equation for the considered problem and it must be solved together with boundary conditions (2), (3).

Equation (5) is a mixed type functional differential equation. Theory of functional differential equations of this type was intensive developed in the last decades. The survey of this theory see in [2]. Theorems of existence and uniqueness of solutions to boundary value problems for mixed functional differential equations were proved in [1], but analytical methods of solution of these boundary value problems are rather complicated and can be applied only in very simple cases. Therefore it is important to develop the approximate methods of solution. The finite differences method was applied to the solution of these type boundary value problems in [3]. Here we investigate the application of direct methods of calculus of variations to these problems for the case of the quadratic functionals.

2 Minimum of the quadratic functional

Consider the problem of the minimum of the functional

$$J(u) = \int\limits_G \left[\sum_{i,j=-r}^{r} \Big(a_{ij}(t,s)\, u_t(t,s+i)\, u_t(t,s+j) + b_{ij}(t,s)\, u(t,s+i)\, u(t,s+j) \Big) + \right.$$

$$\left. + \, 2 \sum_{i=-r}^{r} \Big(d_i(t,s)\, u_t(t,s+i) + e_i(t,s)\, u(t,s+i) \Big) \right] dt\, ds. \tag{6}$$

In the well known way with the use of a change of the unknown function the general boundary conditions can be replaced by the zero boundary conditions. Therefore without loss of generality we can consider functional $J(u)$ under boundary conditions

$$u(t,s) = 0, \quad (t,s) \notin Q. \tag{7}$$

Define the set

$$G = (t_0, t_1) \times (s_0, s_1)$$

and remark that $Q \subset G \subset \mathbb{R}^2$. Here $a_{ij}, d_i \in C^{1,0}(\overline{G})$ are the continuous on the set $\overline{G}$, continuously differentiable in t functions, $b_{ij}, e_i \in C(\overline{G})$ are the continuous on the set $\overline{G}$ functions, $i,j = -r, \dots, r$.

Denote the space H as the subspace of $W^{1,0}(Q)$, which is the closure in $W^{1,0}(Q)$ of the set $\dot{C}^{\infty}(Q)$ of indefinitely smooth functions, finite in Q. The solution of problem (6),(7) we seek in the space H, and the functional $J(u)$ acts from H to $\mathbb{R}$ with the domain $D(J) = H$.

The functional $J(u)$ can be presented in the form

$$J(u) = a(u) - 2\, l(u), \tag{8}$$

where

$$a(u) = \int\limits_G \left[\sum_{i,j=-r}^{r} \Big(a_{ij}(t,s)\, u_t(t,s+i)\, u_t(t,s+j) + b_{ij}(t,s)\, u(t,s+i)\, u(t,s+j) \Big) \right] dt\, ds$$

is a quadratic form and

$$l(u) = - \int\limits_G \left[\sum_{i=-r}^{r} \Big(d_i(t,s)\, u_t(t,s+i) + e_i(t,s)\, u(t,s+i) \Big) \right] dt\, ds$$

is a linear functional.

Introduce the bilinear form $a(u,v)$ that corresponds to the quadratic form $a(u)$

$$a(u,v) = \int\limits_G \left[\sum_{i,j=-r}^{r} \Big(a_{ij}(t,s)\, u_t(t,s+i)\, v_t(t,s+j) + b_{ij}(t,s)\, u(t,s+i)\, v(t,s+j) \Big) \right] dt\, ds.$$

By the change of variables and taking in account boundary condition (7) we get

$$a(u,v) = \int_Q \Bigg[\sum_{i,j=-r}^{r} \Big(a_{ij}(t,s-j) u_t(t,s+i-j) v_t(t,s)$$

$$+ b_{ij}(t,s-j) u(t,s+i-j) v(t,s) \Big) \Bigg] \, dt\, ds.$$

For functions $u \in W^{2,0}(Q) \cap H, v \in H$ we apply the integration by parts and receive

$$a(u,v) = \int_Q \Bigg[\sum_{i,j=-r}^{r} \Big(-(a_{ij}(t,s-j) u_t(t,s+i-j))_t + b_{ij}(t,s-j) u(t,s+i-j) \Big) v(t,s) \Bigg] \, dt\, ds.$$

Denote

$$\bar{a}_k(t,s) = \sum_{\substack{-r \le i,j \le r \\ i-j=k}} a_{ij}(t,s-j), \quad \bar{b}_k(t,s) = \sum_{\substack{-r \le i,j \le r \\ i-j=k}} b_{ij}(t,s-j),$$

and we get for all $u \in W^{2,0}(Q) \cap H$ and $v \in H$

$$a(u,v) = \int\limits_Q \Bigg[\sum_{k=-2r}^{2r} \Big(-(\bar{a}_k(t,s)\, u_t(t,s+k))_t + \bar{b}_k(t,s)\, u(t,s+k) \Big) \Bigg] v(t,s) \, dt\, ds.$$

Define the operator

$$Au = \sum_{k=-2r}^{2r} \Big(-(\bar{a}_k(t,s)\, u_t(t,s+k))_t + \bar{b}_k(t,s)\, u(t,s+k) \Big), \tag{9}$$

$A : L_2(Q) \to L_2(Q)$ with the domain $D(A) = W^{2,0}(Q) \cap H$. We can write

$$a(u,v) = (Au, v) \quad \text{for all } u \in D(A),\ v \in H.$$

Consider the linear functional $l(u)$. In the same way as was done above using integration, boundary condition (7) and integration by parts, we receive for all $u \in H$

$$l(u) = \int\limits_Q \Bigg[\sum_{i=-r}^{r} \Big((d_i(t,s-i))_t - e_i(t,s-i) \Big) u(t,s) \Bigg] \, dt\, ds.$$

Denote

$$f(t,s) = \sum_{i=-r}^{r} \Big((d_i(t,s-i))_t - e_i(t,s-i) \Big), \quad f \in C(\overline{Q}) \subset L_2(Q), \tag{10}$$

and we get

$$l(u) = (u, f) \quad \text{for all } u \in H.$$

It is possible now to present (8) in the form

$$J(u) = (Au, u) - 2(u, f). \tag{11}$$

for all $u \in D(A)$.

Introduce matrices $\mathbb{A}$ and $\mathbb{B}$

$$\mathbb{A}(t,s) = [a_{ij}(t,s)]_{i,j=-r}^{r}, \quad \mathbb{B}(t,s) = [b_{ij}(t,s)]_{i,j=-r}^{r}, \quad (t,s) \in G.$$

Suppose that matrices $\mathbb{A}$, $\mathbb{B}$ are symmetric

$$\mathbb{A}(t,s) = \mathbb{A}^T(t,s), \quad \mathbb{B}(t,s) = \mathbb{B}^T(t,s) \quad \text{for almost all } (t,s) \in G. \tag{12}$$

It is easy to see, that then and the linear operator A is symmetric

$$(Au, v) = (u, Av) \quad \text{for all } u, v \in D(A).$$

Lemma 1 *Suppose that matrices*

$$\mathbb{A}_0(t,s) = \left[a_{ij}(t,s) - \beta^2\,\delta_{0,i}\,\delta_{0,j}\right]_{i,j=-r}^{r}, \quad \mathbb{B}_0(t,s) = \left[b_{ij}(t,s) - \gamma^2\,\delta_{0,i}\,\delta_{0,j}\right]_{i,j=-r}^{r},$$

where $\delta_{0,i}$ *is the Kronecker symbol,* $(t,s) \in G$, *are non-negative definite at some* β, $\gamma \neq 0$

$$x^T \mathbb{A}_0(t,s)\, x \geq 0, \quad x^T \mathbb{B}_0(t,s)\, x \geq 0 \tag{13}$$

for all $x \in \mathbb{R}^{2r+1}$ *at almost all* $(t,s) \in G$. *Then the operator* A *is positive definite*

$$(Au, u) \geq \gamma^2 \|u\|^2 \quad \textit{for all } u \in D(A).$$

Proof Denote

$$x(t,s) = [u(t, s+i)]_{i=-r}^{r}, \quad y(t,s) = [u_t(t, s+i)]_{i=-r}^{r},$$

and we receive for all $u \in D(A)$

$$(Au, u) = \int_G \Big(y^T(t,s)\mathbb{A}(t,s)\, y(t,s) + x^T(t,s)\mathbb{B}(t,s)\, x(t,s) \Big) dt\, ds$$

$$= \int_G \Big(y^T(t,s)\mathbb{A}_0(t,s)\, y(t,s) + x^T(t,s)\mathbb{B}_0(t,s)\, x(t,s) + \beta^2 u_t^2(t,s) + \gamma^2 u^2(t,s) \Big) dt\, ds \geq \gamma^2 \|u\|^2.$$

Remark that if $\beta \neq 0$, then

$$(Au, u) \geq \min(\beta^2, \gamma^2)\, \|u\|_1^2 \quad \text{for all } u \in D(A). \blacksquare$$

3 The Ritz method

Suppose that conditions (12), (13) are fulfilled. Then, as it was proved above, there exists a linear positive definite operator (9) $A : L_2(Q) \to L_2(Q)$ with everywhere dense domain $D(A) = W^{2,0}(Q) \cap H$ and such a function (10) $f \in L_2(Q)$ that the functional (6) can be written in the form (11) for all $u \in D(A)$.

Apply the **Ritz method.** Introduce in $D(A)$ the new scalar product and norm

$$[u, v] = (Au, v), \quad \|u\|_A = [u, u]^{1/2}.$$

Supplement now $D(A)$ using the introduced norm $\|\cdot\|_A$ and we receive the complete space H_A, which is called the **energy space** generated by the operator A. It is known (see [5], Ch.1, §3) that all elements of H_A belong to H.

Lemma 2 *Suppose that conditions (12), (13) are fulfilled for some $\beta \neq 0$. Then spaces H_A and H coincide.*

Proof It is sufficient to establish the equivalence of the norms $\|\cdot\|_1$ and $\|\cdot\|_A$. Then from the density of $D(A)$ in H and of the completeness of H it follows that the closure of $D(A)$ coincides with H.

1. The inequality $\|u\|_1^2 \le k_1^2 \, \|u\|_A^2$. follows from Lemma 1
2. Prove now the inequality $\|u\|_A^2 \le k_2^2 \, \|u\|_1^2$

$$\|u\|_A^2 = (Au, u) = \le \sum_{i,j=-r}^{r} \left| \int_G a_{ij}(t, s) \, u_t(t, s+i) \, u_t(t, s+j) \, dt \, ds \right|$$

$$+ \sum_{i,j=-r}^{r} \left| \int_G b_{ij}(t, s) \, u(t, s+i) \, u(t, s+j) \, dt \, ds \right| \le \sum_{i,j=-r}^{r} p_{ij}^2 \, \|u_t\|^2 + \sum_{i,j=-r}^{r} q_{ij}^2 \, \|u\|^2 \le k_2^2 \|u\|_1^2.$$

■

It is known (see [4], Ch.1, §2) that under our assumptions there exists a unique element $u_0 \in H_A$ such that the functional

$$J(u) = [u, u] - 2(u, f), \tag{14}$$

which is the extension of the functional (11) on H_A, attains a minimum. This element u_0 is called the generalized solution of the problem of minimum for the functional (6). If $u_0 \in D(A)$, then this solution is called the classical solution.

Introduce the system of the linear independent functions $\{\varphi_k\}$ that is complete in H_A. The Ritz approximate solution we construct in the form

$$u_N = \sum_{k=1}^{N} c_k \varphi_k.$$

The unknown coefficients of the expansion $\{c_k\}$ we will define as solution to the system of linear equations

$$[u_N, \varphi_k] = (f, \varphi_k), \quad k = 1, \ldots, N. \tag{15}$$

This system has a unique solution because its determinant is a Gram determinant of the linear independant system of functions $\{\varphi_k\}$. System (15) can be written in the matrix form

$$\tilde{A}c = \tilde{b},$$

where

$$c = (c_1, \ldots, c_N)^T, \quad b = (b_1, \ldots, b_N)^T;$$
$$\tilde{A}_{i,j} = [\varphi_i, \varphi_j], \quad i, j = 1, \ldots, N;$$
$$\tilde{b}_i = \int\limits_G \left[\sum_{i=-r}^{r} \Big(d_i(t,s)\,(\varphi_i)_t(t, s+i) + e_i(t,s)\,\varphi_i(t, s+i)\Big)\right] dt\,ds, \quad i = 1, \ldots, N.$$

Theorem 2 (Miklin [5], Ch.1, §8) *If the operator A is positive definite, then the Ritz successive approximations to the solution of the variational problem converge in the metric of the space H_A to its exact solution u_0.*

From the inequality

$$\|u_N - u_0\| \le \frac{1}{\gamma^2}\|u_N - u_0\|_A \longrightarrow 0$$

it follows that the successive approximations converge to u_0 in the metric of the space H. Suppose that basis functions $\{\varphi_k\}$ belong to $D(A)$ (it is possible because $D(A)$ is dense in H_A. Suppose also that there exists a solution $u_0 \in D(A)$. Then it is true the evaluation of the rate of convergence in the space $L_2(Q)$ ([4], Ch.1, §2)

$$\|u_0 - u_N\|^2 \le \frac{\|A^{-1}\|}{\gamma^2}\|A(u_0 - u_N)\|^2.$$

From the a priori evaluation of the Ritz approximations $\|u_N\| \le \|f\|/\gamma^2$ ([4], Ch.1, §2) and taking in account the convergence to the exact solution $u_0 \in D(A)$ we receive in the limit the evaluation of the norm of the inverse operator

$$\|A^{-1}\| \le 1/\gamma^2. \tag{16}$$

From (16) and from $Au_0 = f$ we receive the evaluation of the rate of convergence in $L_2(Q)$ of the Ritz approximations to the classical solution $u_0 \in D(A)$ in the case $\{\varphi_k\} \subset D(A)$:

$$\|u_0 - u_N\| \le \frac{1}{\gamma^2}\|f - Au_N\|.$$

Consider the element Au_N. Expand it on the introduced system of functions

$$(Au_N, \varphi_i) = \sum_{k=1}^{N} c_k(A\varphi_k, \varphi_i) = \sum_{k=1}^{N} c_k[\varphi_k, \varphi_i],$$

$$Au_N = \sum_{k=1}^{N}\sum_{i=1}^{\infty}\frac{c_k}{\|\varphi_i\|^2}[\varphi_k,\varphi_i]\varphi_i.$$

If the system of functions is orthogonal in H_A, we receive

$$Au_N = \sum_{k=1}^{N} c_k \frac{\|\varphi_k\|_A{}^2}{\|\varphi_k\|^2}\varphi_k.$$

It means that if the system is orthogonal in H_A, then the expansion on the basis functions of the image N–th approximation contains exactly N members of the series

$$Au_N = \tilde{u}_N, \quad \tilde{u}_N = \sum_{k=1}^{N}\tilde{c}_k\varphi_k, \quad \tilde{c}_k = c_k\frac{\|\varphi_k\|_A{}^2}{\|\varphi_k\|^2}.$$

Remark that if the system is orthogonal in H_A, then it is also orthogonal in $L_2(Q)$. Suppose that $i \neq j$. Then

$$A\varphi_i = \frac{\|\varphi_i\|_A^2}{\|\varphi_i\|^2}\varphi_i, \quad 0 = (A\varphi_i,\varphi_j) = \frac{\|\varphi_i\|_A^2}{\|\varphi_i\|^2}(\varphi_i,\varphi_j), \quad (\varphi_i,\varphi_j) = 0.$$

Consider the rate of convergence of the method for the system of functions that is orthogonal in H_A. By the construction of the Ritz method Au_N has the same coefficients of expansion for the first N basis functions as the function f. Therefore

$$f - Au_N = \sum_{i=1}^{\infty} f_i\varphi_i - \sum_{i=1}^{N}\tilde{c}_i\varphi_i = \sum_{i=N+1}^{\infty} f_i\varphi_i$$

Compute the coefficients of the expansion f_k and we receive the evaluation of the rate of convergence for the orthogonal in H_A system of functions.

$$\|u_0 - u_N\| \leq \frac{1}{\gamma^2}\left\|\sum_{i=N+1}^{\infty} f_i\varphi_i\right\| = \frac{1}{\gamma^2}\left\|\sum_{i=N+1}^{\infty}\frac{(\varphi_i,f)}{\|\varphi_i\|^2}\varphi_i\right\|$$

or

$$\|u_0 - u_N\| \leq \frac{1}{\gamma^2}\left\|\sum_{i=N+1}^{\infty}\frac{\varphi_i}{\|\varphi_i\|^2}\int_G l(\varphi_i)dt\,ds\right\|.$$

In the general case for the system of functions that is not orthogonal in H_A in the same way we receive

$$\|u_0 - u_N\| \leq \frac{1}{\gamma^2}\left\|\sum_{i=N+1}^{\infty}\left(f_i\varphi_i - \sum_{k=1}^{N}\frac{c_k}{\|\varphi_i\|^2}[\varphi_k,\varphi_i]\varphi_i\right)\right\|$$

or

$$\|u_0 - u_N\| \leq \frac{1}{\gamma^2}\left\|\sum_{i=N+1}^{\infty}\frac{\varphi_i}{\|\varphi_i\|^2}(\sum_{k=1}^{N} c_k a(\varphi_k,\varphi_i) + l(\varphi_i))\right\|.$$

4 The least squares method

Consider the linear operator $A : L_2(Q) \to L_2(Q)$, $D(A) = W^{2,0} \cap H$

$$Au = \sum_{k=-2r}^{2r} \Big(-(\bar{a}_k(t,s)\, u_t(t,s+k))_t + \bar{b}_k(t,s)\, u(t,s+k) \Big),$$

that was defined before (9). Suppose again that it is symmetric and positive definite, i.e. the conditions (12), (13) are fulfilled.

It is known that (see [4], Ch.1, §2) an element $u_0 \in D(A)$ furnishes a minimum to functional (11) iff this element is a solution to the equation

$$Au = f, \tag{17}$$

where f is defined by (10). In [5], Ch.1, §9 it is proved that a positive definite operator defined on a everywhere dense set can be extended to the selfadjoint operator A. The domain D_A contains all elements $u \in H$, on which functional (14) attains a minimum for any $f \in L_2(Q)$ and is dense in H_A. The scalar product

$$[u, v] = (Au, v)$$

is valid for $u \in D_{\mathcal{A}}$, $v \in H_A$. Therefore the equation

$$Au = f$$

for any $f \in L_2(Q)$ has a solution that furnishes a minimum to functional (14). Apply the **least squares method** for solution of this equation.

Let $\{\psi_k\}$, be a system of linear independent functions such that the system $\{A(\psi_k)\}$ is complete in $L_2(Q)$. It is proved in [5], Ch.1, §9 that for a selfadjoint operator A such a system exists because the space $L_2(Q)$ separable. It is sufficient to take a preimage of a some dense in $L_2(Q)$ sequence of linear independent functions. The system $\{A(\psi_k)\}$ will be also linear independent. The solution we seek in the form

$$u_N = \sum_{k=1}^{N} c_k \psi_k.$$

The coefficients $\{c_k\}$ are defined from the least squares condition

$$(Au_N - f, Au_N - f) \longrightarrow \min$$

From the necessary condition of a minimum for the function of N variables $c_k, k = 1, \ldots, N$ we receive the system of linear equations

$$\sum_{k=1}^{N} c_k\, (A\psi_k, A\psi_i) = (f, A\psi_i), \quad i = 1, \ldots N, \tag{18}$$

or, in the matrix form,

$$\hat{A}c = \hat{b},$$

where

$$\hat{b}_i = (f, A\psi_i), \quad i = 1, \ldots, N,$$
$$\hat{A}_{ij} = (A\psi_i, A\psi_j) \quad i, j = 1, \ldots, N.$$

System (18) has a unique solution because its determinant is the Gram determinant of the linear independent elements $\{A(\psi_k)\}$.

Definition The sequence $\{v_n\}$, $v_n \in H_A$ is called the **minimizing sequence** if

$$\lim_{n\to\infty} F(v_n) = \inf_{u\in H_A} F(u).$$

It is proved in [5], Ch.1, §8 that the sequence u_N is a minimizing sequence. It is also proved there that any minimizing sequence converges in H_A to the element that furnishes a minimum to functional (14). Therefore the approximations u_N that are constructed by the least squares method converge to the element $u_0 \in H_A$ that furnishes a minimum to functional (14) in the metric of the space H_A

$$\|u_N - u_0\|_{H_A} \to 0 \quad N \to \infty,$$

and also in the metric of the space $L_2(Q)$

$$\|u_N - u_0\| \to 0 \quad N \to \infty.$$

From the boundedness of the inverse operator A^{-1} (see [5], Ch.1, §5) it follows the evaluation of the rate of convergence

$$\|u_N - u_0\| < k\|Au_N - f\|.$$

References

[1] G.A. Kamenskii. Boundary value problems for differential–difference equations arising from variational problems. *Nonlinear Analysis*, TMA, **18**(8) (1992), 801–813.

[2] G.A. Kamenski. A review of the theory of mixed functional–difference equations. *Problems of Nonlinear Analysis in Engineering Systems*, **2**(8) (1998), 1–16.

[3] A.F.Kopylov. Method of finite differences of numerical solution of the mixed type differential-difference equation. Sibirian journal of computational mathematics v.3, No 4, 2000, pp.345-355. (In Russian.)

[4] G.I.Marchuk, V.I. Agoshkov. Introduction to projective-net methods. Moscow, Nauka, 1981. (In Russian.)

[5] S.G. Mikhlin. Direct methods in mathematical physics. Moscow, Gostechizdat, 1950. (In Russian.)