FUNCTIONAL DIFFERENTIAL EQUATIONS

VOLUME 2004, NO PP.

ON THE EXISTENCE OF ORTHONORMAL BASIS CONSISTING OF EIGENFUNCTIONS OF ELLIPTIC FUNCTIONAL DIFFERENTIAL OPERATORS*

E. M. VARFOLOMEYEV†

Abstract. The necessary and sufficient conditions for normality of elliptic functional-differential operators with two transformations of variables were obtained. The normality of such an operator is equivalent to the existence of orthonormal basis consisting of eigenfunctions of this operator.

Key Words. Normality, not self-adjoint elliptic functional-differential operator, nonlinear optics.

AMS(MOS) subject classification. 47F05, 47B15, 35Q60.

Introduction. A field transformation in the two-dimensional feedback in a nonlinear optical system leads to multi-petal waves generation [1, 2]. Those light structures arise in contemporary computer technology developing optical analogues of neuron nets. The mathematical model of such a system is described by bifurcation of periodic solutions for quasi-linear parabolic functional differential equation with transformation of spatial variables $g(x)$. In [3, 4] this problem is considered in the case where the spatial domain Q is a circle or a ring and the transformation of spatial variables g is a rotation by some angle θ. An arbitrary domain $Q \subset \mathbb{R}^2$ and an arbitrary transformation g were considered in [5, 6]. In these works the linearized elliptic functional differential operator was assumed to be normal. In [7] necessary and sufficient conditions for the normality were obtained in terms of properties of domain $Q \subset \mathbb{R}^n$ and transformation g. A more general case without the assumption of normality was considered in [8].

* Supported by the Russian Foundation for Basic Research (Grant 04-01-00256) and Russian Ministry of Education (Grant UR.04.01.129).

† Nelidovskaya st. 25k1, flat 72, 125363 Moscow, Russian Federation

It was proved [6] that the normality of the linearized elliptic functional differential operator of this problem is equivalent to the existence of orthonormal basis consisting of eigenfunctions of this operator. This result also is valid for such operator with two transformations of variables.

In this work necessary and sufficient conditions for normality of the linearized operator were obtained in the case of two transformations of spatial variables. Such operator has a system of eigenfunctions forming an orthonormal basis in $L_2(Q)$ if and only if the operator is normal.

1. Formulation of the problem. Let $Q \subset \mathbb{R}^n$ be a bounded domain with the boundary $\partial Q \subset C^\infty$, $n \geq 2$, and let g and f be one-to-one transformations of class C^3 such that

$$g : V \subset \mathbb{R}^n \to g(V) \subset \mathbb{R}^n, \quad |J_g(x)| \neq 0, \; x \in V,$$
$$f : V \subset \mathbb{R}^n \to f(V) \subset \mathbb{R}^n, \quad |J_f(x)| \neq 0, \; x \in V,$$

where V is a bounded domain such that $\overline{Q} \subset V$, $J_g(x) = [\partial g_i/\partial x_j]_{i,j=1}^n$ is the Jacobi matrix of transformation g, and $|J_g(x)| = |\det J_g(x)|$. Also we assume that the following holds:

$$g(Q) \subset Q, \quad f(Q) \subset Q. \tag{1.1}$$

Consider an unbounded operator $A_0 : L_2(Q) \to L_2(Q)$ with the domain $D(A_0) = \{v \in W_2^2(Q) : Bv = 0\}$ defined by the formula $A_0 v = \Delta v$, $v \in D(A_0)$. Here $W_2^k(Q)$ denotes the Sobolev space of complex-valued functions in $L_2(Q)$ such that all of their generalized derivatives up to k-th order belong $L_2(Q)$, $Bv = v|_{\partial Q}$ or $Bv = (\partial v/\partial \nu)|_{\partial Q}$, and ν is a unit vector normal to ∂Q at a point $x \in \partial Q$. It is well-known that A_0 is a self-adjoint operator. Consider an operator $A : L_2(Q) \to L_2(Q)$ such that $A = A_0 + A_1 + A_2$, where A_1 and A_2 are linear bounded operators defined on the whole space $L_2(Q)$ as follows:

$$A_1 : L_2(Q) \to L_2(Q), \quad A_1 v(x) = a_1 v(g(x)),$$
$$A_2 : L_2(Q) \to L_2(Q), \quad A_2 v(x) = a_2 v(f(x)),$$

where $a_1 \neq 0$ and $a_2 \neq 0$ are real numbers.

An operator A is said to be normal if $D(AA^*) = D(A^*A)$ and $AA^*v = A^*Av$ for all $v \in D(A^*A)$. Put $D(A) = D(A_0)$.

Introduce sets $G_g^m = \{x \in Q : g^m(x) \neq x\}$, $m = 1, 2, \ldots$. Here $g^m(x)$ denotes transformation g applied m times. Denote $\widetilde{G}_g^m = Q \setminus G_g^m$. We write a superposition of transformations in the form like $fg(x)$, $g^{-1}f(x)$, etc.

2. The main results.

THEOREM 1. *Suppose that* $G_g^2 \neq \varnothing$ *and* $G_f^2 \neq \varnothing$. *Moreover, let conditions* $g(Q) = f(Q) = Q$ *and* $|a_1| \neq |a_2|$ *hold. Then operator* A *is normal if and only if*

$$g(x) = Kx + b, \quad f(x) = Cx + d, \quad x \in Q, \tag{2.1}$$

$$gf(x) = fg(x), \quad x \in Q, \tag{2.2}$$

where K *and* C *are orthogonal matrices of* $n \times n$ *size such that* $K^2 \neq E$ *and* $C^2 \neq E$, $b, d \in \mathbb{R}^n$.

THEOREM 2. *Suppose that* $G_g^2 = \varnothing$ *and* $G_f^2 = \varnothing$. *Then* $g(Q) = f(Q) = Q$ *and*:

1. *If operator* A *is normal and condition* $a_1 + a_2 \neq 0$ *holds, then*

$$\begin{cases} a_1^2 \left(|J_g(x)| - |J_g(x)|^{-1}\right) + a_2^2 \left(|J_f(x)| - |J_f(x)|^{-1}\right) = 0, & x \in G_g^1 \cap G_f^1, \\ |J_g(x)| = |J_f(x)| = 1, & x \in Q \setminus (G_g^1 \cap G_f^1). \end{cases}$$

2. *If* $|J_g(x)| = |J_f(x)| = 1$ *for any* $x \in Q$, *then* A *is normal and self-adjoint operator.*

THEOREM 3. *Suppose that* $G_g^2 \neq \varnothing$ *and* $G_f^2 = \varnothing$. *Moreover, let* $g(Q) = Q$. *Then* $f(Q) = Q$ *and operator* A *is normal if and only if*

$$\begin{aligned} &g(x) = Kx + b, \quad |J_f(x)| = 1, \quad x \in Q, \\ &gf(x) = fg(x), \quad x \in Q, \end{aligned} \tag{2.3}$$

where K *is an orthogonal matrix of* $n \times n$ *size such that* $K^2 \neq E$ *and* $b \in \mathbb{R}^n$.

3. Some auxiliary statements.

LEMMA 1. *Adjoint operator* A_1^* *is described by the following formula*:

$$A_1^* v(x) = \begin{cases} a_1 |J_{g^{-1}}(x)| v(g^{-1}(x)) & \text{for } x \in g(Q), \\ 0 & \text{for } x \in Q \setminus g(Q), \end{cases} \tag{3.1}$$

where $J_{g^{-1}}(x)$ *is Jacobi matrix for transformation* g^{-1}. The proof is obvious: it is sufficient to change the integration variable in the $L_2(Q)$ scalar product. Adjoint operator A_2^* is described analogously.

REMARK 1. *Since* $D(A_0) = D(A_0^*)$ *and linear operators* $A_k, A_k^* : L_2(Q) \to L_2(Q)$ *are bounded,* $k = 1, 2$, *we have* $D(A) = D(A^*) = D(A_0)$.

LEMMA 2. *Suppose that* $g(Q) = f(Q) = Q$ *and the following conditions hold for any* $x \in Q$:

$$\begin{gathered} |J_{f^{-1}}(x)| = |J_{f^{-1}}(f(x))|, \quad |J_{g^{-1}}(x)| = |J_{g^{-1}}(g(x))|, \\ |J_{f^{-1}}(x)| = |J_{f^{-1}}(g(x))|, \quad |J_{g^{-1}}(x)| = |J_{g^{-1}}(f(x))|, \\ fg(x) = gf(x). \end{gathered}$$

Then operator $A_1 + A_2$ is normal.

Proof. Using Lemma 1, for any function $v(x) \in L_2(Q)$ we get

$$\begin{aligned}
A_1^*A_1v(x) &= a_1^2|J_{g^{-1}}(x)|v(x), & A_2^*A_1v(x) &= a_1a_2|J_{f^{-1}}(x)|v(gf^{-1}(x)),\\
A_2^*A_2v(x) &= a_2^2|J_{f^{-1}}(x)|v(x), & A_1^*A_2v(x) &= a_1a_2|J_{g^{-1}}(x)|v(fg^{-1}(x)),\\
A_1A_1^*v(x) &= a_1^2|J_{g^{-1}}(g(x))|v(x), & A_1A_2^*v(x) &= a_1a_2|J_{f^{-1}}(g(x))|v(f^{-1}g(x)),\\
A_2A_2^*v(x) &= a_2^2|J_{f^{-1}}(f(x))|v(x), & A_2A_1^*v(x) &= a_1a_2|J_{g^{-1}}(f(x))|v(g^{-1}f(x)).
\end{aligned}$$

Conditions $fg(x) = gf(x)$ and $g(Q) = f(Q) = Q$ imply relations $f^{-1}g(x) = gf^{-1}(x)$, $fg^{-1}(x) = g^{-1}f(x)$, and $f^{-1}g^{-1}(x) = g^{-1}f^{-1}(x)$ for all $x \in Q$. Hence, taking into account the conditions imposed on jacobians, we get

$$(A_1 + A_2)(A_1^* + A_2^*)\,v(x) = (A_1^* + A_2^*)(A_1 + A_2)\,v(x), \quad v(x) \in L_2(Q),$$

which proves the lemma. □

REMARK 2. *From the identity $|J_g(x)| \cdot |J_{g^{-1}}(g(x))| = 1$, it follows that the condition $|J_{g^{-1}}(x)| = |J_{g^{-1}}(g(x))|$ $(x \in Q)$ is equivalent to the condition $|J_{g^{-1}}(x)| = |J_g(x)|^{-1}$ $(x \in Q)$. Hence we see that condition $|J_{g^{-1}}(x)| = |J_{g^{-1}}(f(x))|$ $(x \in Q)$ is equivalent to condition $|J_g(x)| = |J_g(f(x))|$ $(x \in Q)$. Additionally using an identity $|J_{g^{-1}}(x)| \cdot |J_g(g^{-1}(x))| = 1$, we see that conditions $|J_{g^p}(x)| = |J_{g^p}(g^q(x))|$, where $p, q \in \{-1, 1\}$ and $x \in Q$, are equivalent. The analogous conclusion is valid for the conditions on $|J_f(x)|$ and $|J_{f^{-1}}(x)|$.*

EXAMPLE 1. *Consider an example demonstrating operator A_1+A_2 being not normal in the case of transformations g and f being non-commutative. Put $Q = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 < 4\}$ and let g and f be rotations around axes x_1 and x_2 respectively:*

$$g(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

$$f(x) = \begin{pmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Put $\varphi = \psi = \pi/3$ and fix a point $x^0 = (0,\ 0,\ 1)^T$. We get (see Fig. 1):

$$\begin{aligned}
f^{-1}g(x^0) &= f^{-1}(0,\ -\sqrt{3}/2,\ 1/2)^T = (\sqrt{3}/4,\ -\sqrt{3}/2,\ 1/4)^T,\\
gf^{-1}(x^0) &= g(\sqrt{3}/2,\ 0,\ 1/2)^T = (\sqrt{3}/2,\ -\sqrt{3}/4,\ 1/4)^T,\\
g^{-1}f(x^0) &= g^{-1}(-\sqrt{3}/2,\ 0,\ 1/2)^T = (-\sqrt{3}/2,\ \sqrt{3}/4,\ 1/4)^T,\\
fg^{-1}(x^0) &= f(0,\ \sqrt{3}/2,\ 1/2)^T = (-\sqrt{3}/4,\ \sqrt{3}/2,\ 1/4)^T.
\end{aligned}$$

We have all the conditions of Lemma 2 except for the commutativity of transformations g and f fulfilled. As it was obtained in the proof of Lemma 2, the normality of operator $A_1 + A_2$ is equivalent to the equality

$$v(f^{-1}g(x)) + v(g^{-1}f(x)) = v(fg^{-1}(x)) + v(gf^{-1}(x)) \tag{3.2}$$

for all $v \in L_2(Q)$ and almost all $x \in Q$. Choose $x = x^0$. It is obvious that there exist functions $v \in L_2(Q)$ not satisfying Eq. (3.2). *Therefore, operator $A_1 + A_2$ is not normal.*

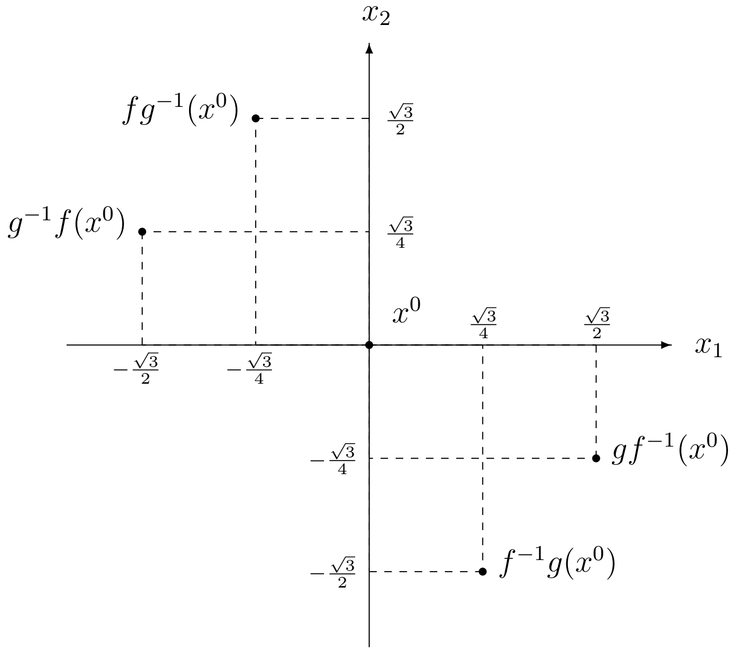

Fig. 1

4. Proof of Theorem 1.

LEMMA 3. *Suppose that $G_g^2 \neq \varnothing$, $G_f^2 \neq \varnothing$, and $g(Q) = f(Q) = Q$. Let operator A be normal and conditions $a_1 a_2 < 0$ and $a_1 + a_2 \neq 0$ hold. Then*

$$g(x) = Kx + b, \quad f(x) = Cx + d, \quad x \in Q.$$

Here K and C are orthogonal matrices of $n \times n$ size such that $K^2 \neq E$ and $C^2 \neq E$, $b, d \in \mathbb{R}^n$.

Proof. We produce the proof for transformation g (transformation f is considered analogously). By the definition, sets G_g^m $(m = 1, 2)$ are open and $G_g^2 \subset G_g^1$. Fix a point $x^0 \in G_g^2$. By the definition of sets G_g^m, for $x = x^0$ the following inequalities take place:

$$(A1) \qquad g(x) \neq x, \qquad\qquad (A2) \qquad g^2(x) \neq x.$$

Since $g(Q) = Q$, it is easy to see that $g(\widetilde{G}_g^2) = \widetilde{G}_g^2$ whence $g(G_g^2) = G_g^2$. Choose some $\delta > 0$ such that $\overline{B_{2\delta}(x^0)} \subset G_g^2$ and the following conditions hold:

$$(B1) \quad B_{2\delta}(x^0) \cap g(B_{2\delta}(x^0)) = \varnothing, \qquad (B2) \quad B_{2\delta}(x^0) \cap g^2(B_{2\delta}(x^0)) = \varnothing.$$

1. Suppose that for $x = x^0$ the following inequalities are valid:

$$\begin{aligned}
&(A3) \quad f(x) \neq x, && (A4) \quad g(x) \neq f(x),\\
&(A5) \quad g(x) \neq f^{-1}(x), && (A6) \quad g^2(x) \neq f(x),\\
&(A7) \quad g(x) \neq fg(x), && (A8) \quad g^{-1}(x) \neq f^{-1}g(x).
\end{aligned}$$

Since transformations g and f are continuous, we can choose $\delta > 0$ small enough to satisfy the following conditions:

$$\begin{aligned}
&(B3) \quad B_{2\delta} \cap f(B_{2\delta}) = \varnothing, && (B4) \quad g(B_{2\delta}) \cap f(B_{2\delta}) = \varnothing,\\
&(B5) \quad g(B_{2\delta}) \cap f^{-1}(B_{2\delta}) = \varnothing, && (B6) \quad g^2(B_{2\delta}) \cap f(B_{2\delta}) = \varnothing,\\
&(B7) \quad g(B_{2\delta}) \cap fg(B_{2\delta}) = \varnothing, && (B8) \quad g^{-1}(B_{2\delta}) \cap f^{-1}g(B_{2\delta}) = \varnothing.
\end{aligned}$$

where $B_{2\delta} = B_{2\delta}(x^0)$.

Now we generalize the method used in [7]. Introduce a function $\xi \in \dot{C}^\infty(\mathbb{R}^n)$ such that $0 \le \xi(x) \le 1$ for all $x \in \mathbb{R}^n$, $\xi(x) = 1$ for all $x \in g(B_\delta(x^0))$, and $\operatorname{supp} \xi \subset g(B_{2\delta}(x^0))$. Put $u = \xi P$, where P is a polynomial. It is obvious that $u \in D(A^*A)$. Consider $AA^*u(x)$ and $A^*Au(x)$. By the definition of function $\xi(x)$ for $x \in B_\delta(x^0)$ we get:

$$\begin{aligned}
&A_0A_2u(x) = 0, && A_2A_0u(x) = 0 && \text{by condition } (B4),\\
&A_0A_1^*u(x) = 0, && A_1^*A_0u(x) = 0 && \text{by condition } (B2),\\
&A_0A_2^*u(x) = 0, && A_2^*A_0u(x) = 0 && \text{by condition } (B5),\\
&A_1A_1^*u(x) = 0, && A_1^*A_1u(x) = 0 && \text{by condition } (B1),\\
&A_2A_2^*u(x) = 0, && A_2^*A_2u(x) = 0 && \text{by condition } (B1),\\
&A_1A_2^*u(x) = 0 && && \text{by condition } (B7),\\
&A_1^*A_2u(x) = 0 && && \text{by condition } (B8),\\
&A_2A_1^*u(x) = 0 && && \text{by condition } (B6),\\
&A_2^*A_1u(x) = 0 && && \text{by condition } (B3).
\end{aligned}$$

Since operator A is normal, we have $AA^*u(x) = A^*Au(x)$ whence

$$A_0A_1u(x) = A_1A_0u(x), \quad x \in B_\delta(x^0).$$

Therefore,

$$\Delta u(g(x)) = (\Delta u)(g(x)), \quad x \in B_\delta(x^0). \tag{4.1}$$

Differentiating composite function $u(g(x))$, from (4.1) we obtain:

$$\sum_{i=1}^{n}\sum_{r,s=1}^{n} u_{g_r g_s}(g(x)) g_{r x_i}(x) g_{s x_i}(x) + \sum_{i=1}^{n}\sum_{r=1}^{n} u_{g_r}(g(x)) g_{r x_i x_i}(x) = \sum_{r=1}^{n} u_{g_r g_r}(g(x)), \quad x \in B_\delta(x^0) \tag{4.2}$$

Put $P(x)$ equal to $(x_k - g_k(x^0))^2$ and $(x_k - g_k(x^0))(x_m - g_m(x^0))$, where $k \neq m$. Then from (4.2) we get:

$$\sum_{i=1}^{n} g_{k x_i}^2(x^0) = 1, \qquad k = 1, \dots, n, \tag{4.3}$$

$$\sum_{i=1}^{n} g_{k x_i}(x^0) g_{m x_i}(x^0) = 0, \qquad k, m = 1, \dots, n, \quad k \neq m. \tag{4.4}$$

Equations (4.3) and (4.4) can be written in a matrix form:

$$J_g(x^0) J_g^T(x^0) = E. \tag{4.5}$$

Therefore,

$$J_g^T(x^0) J_g(x^0) = E. \tag{4.6}$$

Expand (4.6) to a coordinate form:

$$\sum_{i=1}^{n} \frac{\partial g_i(x^0)}{\partial x_k} \frac{\partial g_i(x^0)}{\partial x_m} = \delta_{km}.$$

Differentiating this equation with respect to x_l, we get

$$\sum_{i=1}^{n} \frac{\partial^2 g_i(x^0)}{\partial x_k \partial x_l} \frac{\partial g_i(x^0)}{\partial x_m} + \sum_{i=1}^{n} \frac{\partial g_i(x^0)}{\partial x_k} \frac{\partial^2 g_i(x^0)}{\partial x_l \partial x_m} = 0. \tag{4.7}$$

Cyclically permuting indices k, l, and m in (4.7), we get

$$\sum_{i=1}^{n} \frac{\partial^2 g_i(x^0)}{\partial x_m \partial x_k} \frac{\partial g_i(x^0)}{\partial x_l} + \sum_{i=1}^{n} \frac{\partial g_i(x^0)}{\partial x_m} \frac{\partial^2 g_i(x^0)}{\partial x_k \partial x_l} = 0. \tag{4.8}$$

$$\sum_{i=1}^{n} \frac{\partial^2 g_i(x^0)}{\partial x_l \partial x_m} \frac{\partial g_i(x^0)}{\partial x_k} + \sum_{i=1}^{n} \frac{\partial g_i(x^0)}{\partial x_l} \frac{\partial^2 g_i(x^0)}{\partial x_m \partial x_k} = 0. \tag{4.9}$$

Summing Eq. (4.7) and Eq. (4.8) and subtracting Eq. (4.9), we obtain

$$2 \sum_{i=1}^{n} \frac{\partial^2 g_i(x^0)}{\partial x_k \partial x_l} \frac{\partial g_i(x^0)}{\partial x_m} = 0, \quad k, l, m = 1, \ldots, n.$$

Thus, for any fixed k and l we get a homogeneous system of linear algerbaic equations with the determinant $|J_g(x^0)| \neq 0$. Hence

$$\frac{\partial^2 g_i(x^0)}{\partial x_k \partial x_l} = 0, \quad i, k, l = 1, \ldots, n.$$

Putting $P(x)$ equal to $(x_k - g_k(x^B))(x_m - g_m(x^B))$, where $k, m = 1, \ldots, n$ and x^B is an arbitrary point from the ball $B_\delta(x^0)$, we obtain these equalities for any point $x^B \in B_\delta(x^0)$. Therefore, $g_i(x)$ is a linear function of variables $x_1, \ldots, x_n$, in the neighborhood of point x^0:

$$g(x) = K^{x^0} x + b^{x^0}, \quad x \in B_\delta(x^0). \tag{4.10}$$

By virtue of Eq. (4.5) matrix K^{x^0} is orthogonal.

Now we consider different cases where some of inequalities $(A3)$–$(A8)$ get violated. This leads to certain equalities and, since transformations g and f are smooth, such equalities hold on closed sets. For any boundary point of such sets we can build a sequence of outer points tending to this point. Proceeding to limit, we extend Eq. (4.10) to all boundary points. Below we consider the cases where inequalities $(A3)$–$(A8)$ are violated in closed sets with non-empty interior. Inequalities $(A1)$ and $(A2)$ and conditions $(B1)$ and $(B2)$ remain valid for all cases considered below.

2. Let inequality $(A3)$ be violated in a neighborhood of a point $x^0 \in G_g^2$:

$$f(x) = x, \quad \forall x \in B_{2\delta}(x^0) \subset G_g^2 \tag{$\overline{A3}$}$$

while inequalities $(A4)$–$(A8)$ remain valid. Choose some small $\delta > 0$ such that conditions $(B4)$–$(B8)$ hold and condition $(B3)$ is violated. Introduce a cut-off function $\xi(x)$ on the domain $g(B_{2\delta}(x^0))$ in the same way as in part **1** of the proof. Put $u = \xi P$, where P is a polynomial. Since condition $(B3)$ is violated, we have for $x \in B_\delta(x^0)$

$$A_2^* A_1 u(x) \neq 0.$$

Taking into account ($\overline{A3}$), for $x \in B_\delta(x^0)$ we get

$$A_2^* A_1 u(x) = a_1 a_2 |J_{f^{-1}}(x)| u(gf^{-1}(x)) = a_1 a_2 u(g(x)). \tag{4.11}$$

Since operator A is normal, in the same way as in part **1** for $x \in B_\delta(x^0)$ we get

$$A_0 A_1 u(x) + A_2^* A_1 u(x) = A_1 A_0 u(x).$$

Let $P(x)$ be equal to $(x_k - g_k(x^0))(x_m - g_m(x^0))$, where $k, m = 1, \dots, n$. Since $\xi(x) = 1$ when $x \in B_\delta(x^0)$, we get

$$u(g(x)) = (g_k(x) - g_k(x^0))(g_m(x) - g_m(x^0)),$$
$$\big[u(g(x))\big]_{x_i} = g_{k x_i}(x)(g_m(x) - g_m(x^0)) + g_{m x_i}(x)(g_k(x) - g_k(x^0))$$

whence we obtain

$$u(g(x))\Big|_{x=x^0} = \big[u(g(x))\big]_{x_i}\Big|_{x=x^0} = 0. \tag{4.12}$$

Combining equations (4.11) and (4.12), we get

$$A_2^* A_1 u(x)\big|_{x=x^0} = 0,$$

therefore,

$$A_0 A_1 u(x)\big|_{x=x^0} = A_1 A_0 u(x)\big|_{x=x^0}. \tag{4.13}$$

Putting $P(x)$ equal to $(x_k - g_k(x^B))(x_m - g_m(x^B))$, where $k, m = 1, \dots, n$ and x^B is an arbitrary point from the ball $B_\delta(x^0)$, we obtain Eq. (4.13) for any point $x^B \in B_\delta(x^0)$. Hence we have Eq. (4.1) for all points $x \in B_\delta(x^0)$. Then we get Eq. (4.10) in the same way as in part **1** of the proof.

3. Let inequality ($A4$) be violated in a neighborhood of a point $x^0 \in G_g^2$:

$$g(x) = f(x), \quad \forall x \in B_{2\delta}(x^0) \subset G_g^2 \tag{$\overline{A4}$}$$

while inequalities ($A3$) and ($A5$)–($A8$) remain valid. Choose some small $\delta > 0$ such that conditions ($B3$) and ($B5$)–($B8$) hold and condition ($B4$) is violated. Introduce a cut-off function $\xi(x)$ on the domain $g(B_{2\delta}(x^0))$ in the same way as in part **1** of the proof. Put $u = \xi P$, where P is a polynomial. Since condition ($B4$) is violated, we have for $x \in B_\delta(x^0)$

$$A_0 A_2 u(x) \neq 0, \qquad A_2 A_0 u(x) \neq 0.$$

Taking into account $(\overline{A4})$, for $x \in B_\delta(x^0)$ we get

$$A_0A_2u(x) = a_2\Delta u(f(x)) = a_2\Delta u(g(x)),$$
$$A_2A_0u(x) = a_2(\Delta u)(f(x)) = a_2(\Delta u)(g(x)).$$

Since operator A is normal, in the same way as in part **1** of the proof for $x \in B_\delta(x^0)$ we get

$$A_0A_1u(x) + A_0A_2u(x) = A_1A_0u(x) + A_2A_0u(x),$$
$$(a_1 + a_2)\Delta u(g(x)) = (a_1 + a_2)(\Delta u)(g(x)).$$

Since we postulated that $a_1+a_2 \neq 0$, we have (4.1). Then we obtain Eq. (4.10) in the same way as in part **1** of the proof.

4. Let inequality $(A5)$ be violated in a neighborhood of a point $x^0 \in G_g^2$:

$$(\overline{A5}) \qquad g(x) = f^{-1}(x), \quad \forall x \in B_{2\delta}(x^0) \subset G_g^2$$

while inequalities $(A3)$–$(A4)$ and $(A6)$–$(A8)$ remain valid. Choose some small $\delta > 0$ such that conditions $(B3)$–$(B4)$ and $(B6)$–$(B8)$ hold and condition $(B5)$ is violated. Introduce a cut-off function $\xi(x)$ on the domain $g(B_{2\delta}(x^0))$ in the same way as in part **1** of the proof. Put $u = \xi P$, where P is a polynomial. Since condition $(B5)$ is violated, for $x \in B_\delta(x^0)$ we get

$$A_0A_2^*u(x) \neq 0, \qquad A_2^*A_0u(x) \neq 0.$$

Taking into account $(\overline{A5})$, for $x \in B_\delta(x^0)$ we get

$$A_0A_2^*u(x) = \Delta\big[a_2|J_{f^{-1}}(x)|u(f^{-1}(x))\big] = \Delta\big[a_2|J_g(x)|u(g(x))\big],$$
$$A_2^*A_0u(x) = a_2|J_{f^{-1}}(x)|(\Delta u)(f^{-1}(x)) = a_2|J_{g(x)}|(\Delta u)(g(x)).$$

Since operator A is normal, in the same way as in part **1** of the proof for $x \in B_\delta(x^0)$ we get

$$A_0A_1u(x) + A_2^*A_0u(x) = A_1A_0u(x) + A_0A_2^*u(x),$$
$$a_1\Delta u(g(x)) + a_2|J_g(x)|(\Delta u)(g(x)) = a_1(\Delta u)(g(x)) + \Delta\big[a_2|J_g(x)|u(g(x))\big].$$

Applying the rule

$$\Delta[v(x)w(x)] = \Delta v(x)w(x) + 2(\nabla v(x), \nabla w(x)) + v(x)\Delta w(x)$$

and grouping terms, for $x \in B_\delta(x^0)$ we get

$$\Delta u(g(x))\Big(a_1 - a_2|J_g(x)|\Big) - \sum_{i=1}^{n}\Big[2a_2|J_g(x)|_{x_i}\big[u(g(x))\big]_{x_i} + a_2|J_g(x)|_{x_ix_i}u(g(x))\Big] = (\Delta u)(g(x))\Big(a_1 - a_2|J_g(x)|\Big). \tag{4.14}$$

Let $P(x)$ be equal to $(x_k - g_k(x^0))(x_m - g_m(x^0))$, where $k, m = 1, \dots, n$. Since $\xi(x) = 1$ when $x \in B_\delta(x^0)$, we get Eq. (4.12). Therefore, at the point $x = x^0$ Eq. (4.14) takes form

$$\Big[\Delta u(g(x))\Big(a_1 - a_2|J_g(x)|\Big)\Big]\Big|_{x=x^0} = \Big[(\Delta u)(g(x))\Big(a_1 - a_2|J_g(x)|\Big)\Big]\Big|_{x=x^0}. \tag{4.15}$$

Since condition $a_1a_2 < 0$ holds under assumptions of the lemma, we have $a_1 - a_2|J_g(x)| \neq 0$. Therefore,

$$\Delta u(g(x))\Big|_{x=x^0} = (\Delta u)(g(x))\Big|_{x=x^0}.$$

Putting $P(x)$ equal to $(x_k - g_k(x^B))(x_m - g_m(x^B))$, where $k, m = 1, \dots, n$ and x^B is an arbitrary point from the ball $B_\delta(x^0)$, we obtain the last equation for any point $x^B \in B_\delta(x^0)$. Hence we have Eq. (4.1) for any point $x \in B_\delta(x^0)$. Then we get Eq. (4.10) in the same way as in part **1** of the proof.

5. Let inequality $(A6)$ be violated in a neighborhood of a point $x^0 \in G_g^2$:

$$g^2(x) = f(x), \quad \forall x \in B_{2\delta}(x^0) \subset G_g^2 \tag{$\overline{A6}$}$$

while inequalities $(A3)$–$(A5)$ and $(A7)$–$(A8)$ remain valid. Choose some small $\delta > 0$ such that conditions $(B3)$–$(B5)$ and $(B7)$–$(B8)$ hold and condition $(B6)$ is violated. Introduce a cut-off function $\xi(x)$ on the ball $g(B_{2\delta}(x^0))$ in the same way as in part **1** of the proof. Put $u = \xi P$, where P is a polynomial. Since condition $(B6)$ is violated, for $x \in B_\delta(x^0)$ we get

$$A_2A_1^*u(x) \neq 0.$$

Taking into account $(\overline{A6})$, for $x \in B_\delta(x^0)$ we get

$$A_2A_1^*u(x) = a_1a_2|J_{g^{-1}}(f(x))|u(g^{-1}f(x)) = a_1a_2|J_{g^{-1}}(f(x))|u(g(x)). \tag{4.16}$$

Since operator A is normal, in the same way as in part **1** of the proof for $x \in B_\delta(x^0)$ we get

$$A_0A_1u(x) = A_1A_0u(x) + A_2A_1^*u(x).$$

Let $P(x)$ be equal to $(x_k - g_k(x^0))(x_m - g_m(x^0))$, where $k, m = 1, \dots, n$. Since $\xi(x) = 1$ when $x \in B_\delta(x^0)$, from Eq. (4.12) and Eq. (4.16) we get

$$A_2 A_1^* u(x)\big|_{x=x^0} = 0.$$

Combining two last equations, we obtain Eq. (4.13). Putting $P(x)$ equal to $(x_k - g_k(x^B))(x_m - g_m(x^B))$, where $k, m = 1, \dots, n$ and x^B is an arbitrary point from the ball $B_\delta(x^0)$, we obtain Eq. (4.13) for any point $x^B \in B_\delta(x^0)$. Hence we have Eq. (4.1) for all points $x \in B_\delta(x^0)$. Then we get Eq. (4.10) in the same way as in part **1** of the proof.

6. Let inequality $(A7)$ be violated in a neighborhood of a point $x^0 \in G_g^2$:

$$(\overline{A7}) \qquad g(x) = fg(x), \quad \forall x \in B_{2\delta}(x^0) \subset G_g^2$$

while inequalities $(A3)$–$(A6)$ and $(A8)$ remain valid. Choose some small $\delta > 0$ such that conditions $(B3)$–$(B6)$ and $(B8)$ remain valid and condition $(B7)$ is violated. Introduce a cut-off function $\xi(x)$ on the ball $g(B_{2\delta}(x^0))$ in the same way as in part **1** of the proof. Put $u = \xi P$, where P is a polynomial. Since condition $(B7)$ is violated, for $x \in B_\delta(x^0)$ we get

$$A_1 A_2^* u(x) \neq 0.$$

Taking into account $(\overline{A7})$, for $x \in B_\delta(x^0)$ we get

$$(4.17) \quad A_1 A_2^* u(x) = a_1 a_2 |J_{f^{-1}}(g(x))| u(f^{-1} g(x)) = a_1 a_2 |J_{f^{-1}}(g(x))| u(g(x)).$$

Since operator A is normal, in the same way as in part **1** of the proof for $x \in B_\delta(x^0)$ we get

$$A_0 A_1 u(x) = A_1 A_0 u(x) + A_1 A_2^* u(x).$$

Let $P(x)$ be equal to $(x_k - g_k(x^0))(x_m - g_m(x^0))$, where $k, m = 1, \dots, n$. Since $\xi(x) = 1$ when $x \in B_\delta(x^0)$, from Eq. (4.12) and Eq. (4.17) we get

$$A_1 A_2^* u(x)\big|_{x=x^0} = 0.$$

Combining two last equations, we obtain (4.13). Putting $P(x)$ equal to $(x_k - g_k(x^B))(x_m - g_m(x^B))$, where $k, m = 1, \dots, n$ and x^B is an arbitrary point from the ball $B_\delta(x^0)$, we obtain Eq. (4.13) for any point $x^B \in B_\delta(x^0)$. Hence we have Eq. (4.1) for all points $x \in B_\delta(x^0)$. Then we get Eq. (4.10) in the same way as in part **1** of the proof.

7. Let inequality $(A8)$ be violated in a neighborhood of a point $x^0 \in G_g^2$:

$$(\overline{A8}) \qquad g^{-1}(x) = f^{-1}g(x), \quad \forall x \in B_{2\delta}(x^0) \subset G_g^2$$

while inequalities $(A3)$–$(A7)$ remain valid. Choose some small $\delta > 0$ such that conditions $(B3)$–$(B7)$ remain valid and condition $(B8)$ is violated. Introduce a cut-off function $\xi(x)$ on the ball $g(B_{2\delta}(x^0))$ in the same way as in part **1** of the proof. Put $u = \xi P$, where P is a polynomial. Since condition $(B8)$ is violated, for $x \in B_\delta(x^0)$ we get

$$A_1^* A_2 u(x) \neq 0.$$

Taking into account $(\overline{A8})$, for $x \in B_\delta(x^0)$ we get

$$A_1^* A_2 u(x) = a_1 a_2 |J_{g^{-1}}(x)| u(fg^{-1}(x)) = a_1 a_2 |J_{g^{-1}}(x)| u(g(x)). \tag{4.18}$$

Since operator A is normal, in the same way as in part **1** of the proof for $x \in B_\delta(x^0)$ we get

$$A_0 A_1 u(x) + A_1^* A_2 u(x) = A_1 A_0 u(x).$$

Let $P(x)$ be equal to $(x_k - g_k(x^0))(x_m - g_m(x^0))$, where $k, m = 1, \dots, n$. Since $\xi(x) = 1$ when $x \in B_\delta(x^0)$, from Eq. (4.12) and Eq. (4.18) we get

$$A_1^* A_2 u(x)\big|_{x=x^0} = 0.$$

Combining two last equations, we obtain (4.13). Putting $P(x)$ equal to $(x_k - g_k(x^B))(x_m - g_m(x^B))$, where $k, m = 1, \dots, n$ and x^B is an arbitrary point from the ball $B_\delta(x^0)$, we obtain Eq. (4.13) for any point $x^B \in B_\delta(x^0)$. Hence we have Eq. (4.1) for all points $x \in B_\delta(x^0)$. Then we get Eq. (4.10) in the same way as in part **1** of the proof.

8. Now we consider the relation between the following group properties of transformations g and f:

$(\overline{A3})$	$f(x) = x,$	$(\overline{A4})$	$g(x) = f(x),$
$(\overline{A5})$	$g(x) = f^{-1}(x),$	$(\overline{A6})$	$g^2(x) = f(x),$
$(\overline{A7})$	$g(x) = fg(x),$	$(\overline{A8})$	$g^{-1}(x) = f^{-1}g(x)$

for all $x \in B_{2\delta}(x^0) \subset G_g^2$ (condition $(A2)$ holds).

Example 3 given below illustrates all possible combinations of properties $(\overline{A3})$–$(\overline{A8})$ under the assumption that property $(A2)$ holds. Fig. 2 shows that there exist transformations g and f satisfying any combination of properties $(\overline{A5})$, $(\overline{A6})$, and $(\overline{A8})$ including the combination where none of these properties hold. Property $(\overline{A4})$ is not consistent either with property $(\overline{A6})$ or $(\overline{A8})$ because otherwise property $(A2)$ would be violated. Properties $(\overline{A4})$ and $(\overline{A5})$ can be combined (see Fig. 3) leading to property $f^2(x) = x$. Property $(\overline{A7})$ is not consistent either with property $(\overline{A4})$, $(\overline{A5})$, or $(\overline{A8})$ because otherwise property $(A2)$ would be violated. Properties $(\overline{A7})$ and $(\overline{A6})$ can be combined as shown in Fig. 4. Property $(\overline{A3})$ is not consistent either with property $(\overline{A4})$, $(\overline{A5})$, or $(\overline{A6})$ because otherwise property $(A2)$ would be violated. Properties $(\overline{A3})$ and $(\overline{A7})$ can be combined (see Fig. 5) as well as properties $(\overline{A3})$ and $(\overline{A8})$ (see Fig. 6). Thus, the only possible cases are the following: any combinations (including empty one) of $(\overline{A5})$, $(\overline{A6})$, and $(\overline{A8})$; $(\overline{A4})$ and $(\overline{A5})$; $(\overline{A6})$ and $(\overline{A7})$; $(\overline{A3})$ and $(\overline{A7})$; finally, $(\overline{A3})$ and $(\overline{A8})$.

9. Let some combination of properties $(\overline{A3})$–$(\overline{A8})$ hold in a neighborhood of a point $x^0 \in G_g^2$. Suppose that this combination is not $(\overline{A4})$–$(\overline{A5})$. In that case, combining corresponding parts of the proof **2** and **5**–**7**, we analogously obtain Eq. (4.13). Putting $P(x)$ equal to $(x_k - g_k(x^B))(x_m - g_m(x^B))$, where $k, m = 1, \dots, n$ and x^B is an arbitrary point from $B_\delta(x^0)$, we establish Eq. (4.13) for any point $x^B \in B_\delta(x^0)$. Hence we obtain Eq. (4.1) for any point $x \in B_\delta(x^0)$. Then we obtain Eq. (4.10) in the same way as in part **1** of the proof. Therefore, if conditions $(A4)$ and $(A5)$ hold in a neighborhood of a point $x^0 \in G_g^2$, then transformation g has form (4.10) in this neighborhood, in particular, $|J_g(x)| = 1$.

10. Let inequalities $(A4)$ and $(A5)$ be violated in a neighborhood of a point $x^0 \in G_g^2$:

$$(\overline{A4}) \qquad g(x) = f(x), \quad \forall x \in B_{2\delta}(x^0) \subset G_g^2,$$

$$(\overline{A5}) \qquad g(x) = f^{-1}(x), \quad \forall x \in B_{2\delta}(x^0) \subset G_g^2$$

while inequalities $(A3)$ and $(A6)$–$(A8)$ remain valid. Choose some small $\delta > 0$ such that conditions $(B3)$ and $(B6)$–$(B8)$ remain valid and conditions $(B4)$ and $(B5)$ are violated. Introduce a cut-off function $\xi(x)$ on the ball $g(B_{2\delta}(x^0))$ in the same way as in part **1** of the proof. Put $u = \xi P$, where P is a polynomial. Since conditions $(B4)$ and $(B5)$ are violated, for $x \in B_\delta(x^0)$ we get

$$A_0 A_2 u(x) \neq 0, \quad A_2 A_0 u(x) \neq 0,$$
$$A_0 A_2^* u(x) \neq 0, \quad A_2^* A_0 u(x) \neq 0.$$

Since operator A is normal, in the same way as in part **1** of the proof for $x \in B_\delta(x^0)$ we get

$$A_0 A_1 u(x) + A_0 A_2 u(x) + A_2^* A_0 u(x) = A_1 A_0 u(x) + A_2 A_0 u(x) + A_0 A_2^* u(x).$$

Combining the reductions made in parts **3** and **4** of the proof, from this equation we get

$$\Big[\Delta u(g(x))\Big(a_1 + a_2 - a_2|J_g(x)|\Big)\Big]\Big|_{x=x^0} = \Big[(\Delta u)(g(x))\Big(a_1 + a_2 - a_2|J_g(x)|\Big)\Big]\Big|_{x=x^0}. \tag{4.19}$$

This implies two cases: either Eq. (4.1) holds for $x = x^0$ or, when $a_1/a_2 > -1$, we get

$$|J_g(x)|\Big|_{x=x^0} = \frac{a_1}{a_2} + 1. \tag{4.20}$$

Under assumptions of the lemma we have $a_1 a_2 < 0$ and $a_1 + a_2 \neq 0$ but a set $\{(a_1, a_2) : a_1/a_2 > -1,\ a_1 a_2 < 0,\ a_1 + a_2 \neq 0\}$ is non-empty, therefore, it is possible that Eq. (4.20) holds while Eq. (4.1) does not hold.[1] Putting $P(x)$ equal to $(x_k - g_k(x^B))(x_m - g_m(x^B))$, where $k, m = 1, \ldots, n$ and x^B is an arbitrary point from the ball $B_\delta(x^0)$, we obtain Eq. (4.19) for any point $x^B \in B_\delta(x^0)$. We claim that Eq. (4.20) actually does not hold on any subset of $B_\delta(x^0)$ with non-empty interior.

Indeed, suppose that Eq. (4.20) holds for any $x \in B_\delta(x^0)$. By definition, $B_{2\delta}(x^0) \subset G_g^2$. Properties $(\overline{A4})$ and $(\overline{A5})$ imply $f(x) = f^{-1}(x)$ for $x \in B_{2\delta}(x^0)$, therefore, $B_{2\delta}(x^0) \subset \widetilde{G}_f^2$. By the definition of sets G_g^2 and $\widetilde{G}_f^2$ and by virtue of $(\overline{A4})$, we have $g(B_{2\delta}(x^0)) \subset G_g^2$ and $g(B_{2\delta}(x^0)) \subset \widetilde{G}_f^2$. Properties $(\overline{A4})$ and $(\overline{A5})$ imply $g^{-1}(x) = f(x)$ and $g^{-1}(x) = f^{-1}(x)$ for $x \in g(B_{2\delta}(x^0))$. We claim that $g(B_{2\delta}(x^0)) \cap B_{2\delta}(x^0) = \varnothing$. Indeed, assuming the contrary we get $g(x) = f^{-1}(x)$ and $g^{-1}(x) = f(x)$ for $x \in g(B_{2\delta}(x^0)) \cap B_{2\delta}(x^0)$. Since $g(B_{2\delta}(x^0)) \cap B_{2\delta}(x^0) \subset \widetilde{G}_f^2$, for $x \in g(B_{2\delta}(x^0)) \cap B_{2\delta}(x^0)$ we have also $f(x) = f^{-1}(x)$ whence $g(x) = g^{-1}(x)$. This contradicts relation $g(B_{2\delta}(x^0)) \cap B_{2\delta}(x^0) \subset G_g^2$ and proves that $g(B_{2\delta}(x^0)) \cap B_{2\delta}(x^0) = \varnothing$. Combining $g^{-1}(x) = f(x)$, $g^{-1}(x) = f^{-1}(x)$, and $g(x) \neq g^{-1}(x)$, we get

[1] If we impose a restriction $a_1/a_2 \leq -1$, then we also should require $a_2/a_1 \leq -1$ because it is needed to consider transformation f analogously. But the set $\{(a_1, a_2) : a_1/a_2 \leq -1,\ a_2/a_1 \leq -1,\ a_1 a_2 < 0,\ a_1 + a_2 \neq 0\}$ is empty.

$g(x) \neq f^{-1}(x)$ and $g(x) \neq f(x)$ for $x \in g(B_{2\delta}(x^0))$. Therefore, according to part **9** of the proof, $|J_g(x)| = 1$ for $x \in g(B_{2\delta}(x^0))$. Suppose that Eq. (4.20) holds, i.e., $|J_g(x))| = 1 + a_1/a_2$ for $x \in B_\delta(x^0)$. Then, using the well-known Jacobian relation, we get $|J_{g^{-1}}(x)| = \frac{a_2}{a_1+a_2}$ for $x \in g(B_\delta(x^0))$. This means that $|J_f(x)| = |J_{f^{-1}}(x)| = \frac{a_2}{a_1+a_2}$ for $x \in g(B_\delta(x^0))$. Choose a point $x^1 \in g(B_\delta(x^0))$ and a ball $B_{2\varepsilon}(x^1) \subset g(B_\delta(x^0))$ such that $g(B_{2\varepsilon}(x^1)) \cap B_{2\varepsilon}(x^1) = \varnothing$, $g^{-1}(B_{2\varepsilon}(x^1)) \cap B_{2\varepsilon}(x^1) = \varnothing$, and $g(B_{2\varepsilon}(x^1)) \cap f(B_{2\varepsilon}(x^1)) = \varnothing$. Introduce a cut-off function $\eta \in \dot{C}^\infty(Q)$ such that $0 \le \eta(x) \le 1$ for $x \in Q$, $\eta(x) = 1$ for $x \in B_\varepsilon(x^1)$, and $\operatorname{supp} \eta \subset B_{2\varepsilon}(x^1)$. Since operator A is normal, we have $AA^*\eta = A^*A\eta$. Taking into account that $g(x) \neq g^{-1}(x) = f^{-1}(x) = f(x)$ for $x \in B_{2\varepsilon}(x^1)$, by definitions of $B_{2\varepsilon}(x^1)$ and $\eta(x)$ for $x \in B_\varepsilon(x^1)$ we get:

$$
\begin{aligned}
&A_0A_1\eta(x) = \Delta[a_1\eta(g(x))] = 0, \quad && A_1A_0\eta(x) = a_1(\Delta\eta)(g(x)) = 0,\\
&A_0A_2\eta(x) = \Delta[a_2\eta(f(x))] = 0, \quad && A_2A_0\eta(x) = a_2(\Delta\eta)(f(x)) = 0,\\
&A_1A_1^*\eta(x) = a_1^2|J_{g^{-1}}(g(x))|, && A_2A_2^*\eta(x) = a_2^2|J_{f^{-1}}(f(x))|,\\
&A_1^*A_1\eta(x) = a_1^2|J_{g^{-1}}(x)|, && A_2^*A_2\eta(x) = a_2^2|J_{f^{-1}}(x)| = a_2^2|J_{g^{-1}}(x)|,\\
&A_0A_1^*\eta(x) = \Delta[a_1|J_{g^{-1}}(x)|\eta(g^{-1}(x))] = 0,\\
&A_0A_2^*\eta(x) = \Delta[a_2|J_{f^{-1}}(x)|\eta(f^{-1}(x))] = 0,\\
&A_1^*A_0\eta(x) = a_1|J_{g^{-1}}(x)|(\Delta\eta)(g^{-1}(x)) = 0,\\
&A_2^*A_0\eta(x) = a_2|J_{f^{-1}}(x)|(\Delta\eta)(f^{-1}(x)) = 0,\\
&A_1A_2^*\eta(x) = a_1a_2|J_{f^{-1}}(g(x))|\eta(f^{-1}g(x)) = 0,\\
&A_2A_1^*\eta(x) = a_1a_2|J_{g^{-1}}(f(x))|\eta(g^{-1}f(x)) = 0,\\
&A_1^*A_2\eta(x) = a_1a_2|J_{g^{-1}}(x)|\eta(fg^{-1}(x)) = a_1a_2|J_{g^{-1}}(x)|,\\
&A_2^*A_1\eta(x) = a_1a_2|J_{f^{-1}}(x)|\eta(gf^{-1}(x)) = a_1a_2|J_{f^{-1}}(x)| = a_1a_2|J_{g^{-1}}(x)|.
\end{aligned}
$$

Combining these equations with $AA^*\eta = A^*A\eta$ and using property $|J_g(x)| \cdot |J_{g^{-1}}(g(x))| = 1$ (the same for f) and relations $g^{-1}(x) = f^{-1}(x) = f(x)$, for $x \in B_\varepsilon(x^1)$ we get

$$
\begin{aligned}
a_1^2&|J_{g^{-1}}(g(x))| + a_2^2|J_{f^{-1}}(f(x))|\\
&= a_1^2|J_{g^{-1}}(x)| + a_2^2|J_{f^{-1}}(x)| + a_1a_2|J_{g^{-1}}(x)| + a_1a_2|J_{f^{-1}}(x)|,
\end{aligned}
$$

whence $a_1^2|J_g(x)|^{-1} + a_2^2|J_f(x)|^{-1} = (a_1+a_2)^2|J_f(x)|$. Inserting the values of the jacobians, we obtain

$$
a_1^2 + a_2^2\left(1 + \frac{a_1}{a_2}\right) = (a_1+a_2)^2\frac{a_2}{a_1+a_2}
$$

which yields $a_1 = 0$. This contradicts the initial assumptions and proves that Eq. (4.20) actually does not hold.

Therefore, Eq. (4.1) holds for $x = x^0$. Putting $P(x)$ equal to $(x_k - g_k(x^B))(x_m - g_m(x^B))$, where $k, m = 1, \dots, n$ and x^B is an arbitrary point from the ball $B_\delta(x^0)$, we obtain Eq. (4.13) for any point $x^B \in B_\delta(x^0)$. Hence we have Eq. (4.1) for all points $x \in B_\delta(x^0)$. Then we get Eq. (4.10) in the same way as in part **1** of the proof.

11. In parts **2–10** it is proved that under conditions of the lemma Eq. (4.10) holds for $B_\delta(x^0) \subset G_g^2$ without any additional assumptions. Therefore,

$$g(x) = K_j x + b^j, \quad x \in G_g^{2j}, \tag{4.21}$$

where G_g^{2j} is an open connected component of set G_g^2.

Since $g(Q) = Q$, we have $g(G_g^2) = G_g^2$. Therefore, if $x \in G_g^{2j}$, then $g(x) \in G_g^{2m}$ for some $m = m(j)$. Moreover, since set G_g^{2j} is connected, index m is independent of x. Thus,

$$g^2(x) = K_m K_j x + K_m b^j + b^m, \quad x \in G_g^{2j}. \tag{4.22}$$

First, suppose that $G_g^2 = Q$. Then j takes on the only value $j = 1$. Suppose that $K_1^2 = E$. Then $g^2(x) = x + K_1 b^1 + b^1$ for $x \in Q$. Hence $K_1 b^1 + b^1 = 0$. Therefore, $g^2(x) = x$ for $x \in Q$. This contradicts condition $G_g^2 = Q$. Thus, if $G_g^2 = Q$, then $g(x)$ has form (2.1) where $K = K_1$ and $K^2 \neq E$.

Now suppose that $\widetilde{G}_g^2 \neq \varnothing$. Then $\partial G_g^2 \cap Q = \partial \widetilde{G}_g^2 \cap Q$. Consider a set $\partial G_g^2 \cap Q$. Choose some $z \in \partial G_g^{2j} \cap Q$. Proceeding Eq. (4.22) to limit as $x \to z$ ($x \in G_g^{2j}$), we get

$$K_m K_j z + K_m b^j + b^m = z. \tag{4.23}$$

If $K_m K_j = E$, then $K_m b^j + b^m = 0$. Hence $g^2(x) = x$ for $x \in G_g^{2j}$. This contradicts the definition of G_g^{2j}. Therefore, the set $\partial G_g^{2j} \cap Q$ belongs to a hyperplane of dimension $r \leq n-1$, where r is an order of eigenvalue $\lambda = 1$ of matrix $K_m K_j \neq E$. (When $r = n$ we would get $K_m K_j = E$, because $K_m K_j$ is orthogonal.) If $\lambda = 1$ is not an eigenvalue of matrix $K_m K_j$, then the set $\partial G_g^{2j} \cap Q$ consists of one point. As we supposed, $g \in C^3$. At the same time, $g^2(x)$ is a piecewise-linear function in Q. Therefore, $\widetilde{G}_g^2 \subset \partial G_g^2$. Thus, $g(x)$ also is a piecewise-linear function in Q. Therefore, $g(x)$ has form (2.1) for $x \in Q$. Moreover, since $K_j = K_m = K$, we get $r \leq n-2$ and the set G_g^2 consists of one connected component.[2] □

---

[2] We prove that $r \leq n-2$ as follows. Let matrix K have spectrum $\sigma(K) = \bigcup_{i=1}^{n} \{\lambda_i\}$,

EXAMPLE 2. *Condition $a_1 + a_2 \neq 0$ is essential in this lemma. Indeed, if we have $a_1 + a_2 = 0$, then any (nonlinear) functions $g(x) = f(x)$, $x \in Q$, produce a normal operator $Au(x) = \Delta u(x)$.*

EXAMPLE 3. *Let us consider examples for all possible combinations of properties $(\overline{A3})$–$(\overline{A8})$.*

Transformations g and f illustrated in figures 2–6 are constructed as follows. They transform a unit circle $Q = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$ by a quasi-rotation:

$$g : (x_1, x_2) \mapsto (y_1, y_2), \qquad \begin{cases} y_1 = r\cos(\hat{g}(r,\varphi)) \\ y_2 = r\sin(\hat{g}(r,\varphi)) \end{cases},$$
$$\hat{g}(r,\varphi) = \eta(r)\varphi + (1 - \eta(r))g(\varphi).$$

Here (x_1, x_2) and (y_1, y_2) are rectangular coordinates on the unit circle before and after transformation g; r and φ are polar coordinates corresponding to rectangular coordinates (x_1, x_2); $\eta \in \dot{C}^\infty(\mathbb{R}^+)$ is a strictly decreasing cut-off function such that $0 \leq \eta(r) \leq 1$ for any $r \in \mathbb{R}^+$, $\eta(r) = 1$ for $r \in (0, \varepsilon)$, and $\operatorname{supp}\eta \subset (0, 2\varepsilon)$, where ε is some sufficiently small number. We choose strictly increasing functions $g(\varphi)$ (for brevity denoting this function by the same symbol as transformation g) such that $g \in C^3([0, 2\pi])$ and $g'(\varphi) > 0$, $\varphi \in [0, 2\pi]$. Transformation f is defined analogously. Figures 2–6 show graphs of functions $g(\varphi)$ and $f(\varphi)$ represented modulo 2π (the functions themselves are C^3-smooth). These functions perform a one-to-one mapping of a polar angle φ. Signs "•" on the graphs denote ε-neighborhoods where parts of graphs are coupled C^3-smoothly by strictly increasing functions.

On the whole, we have $g, f \in C^3(Q)$ for the transformations introduced. Indeed, when $r \geq \varepsilon$, taking into account that $\frac{\partial}{\partial x_1} = \cos(\varphi)\frac{\partial}{\partial r} - \frac{\sin(\varphi)}{r}\frac{\partial}{\partial \varphi}$, $\frac{\partial}{\partial x_2} = \sin(\varphi)\frac{\partial}{\partial r} + \frac{\cos(\varphi)}{r}\frac{\partial}{\partial \varphi}$, and $\hat{g}(r,\varphi) \in C^3([0,1] \times [0, 2\pi])$, we obtain $g, f \in C^3([\varepsilon, 1] \times [0, 2\pi])$. When $r < \varepsilon$, we have $\hat{g}(r,\varphi) = \varphi$ (whereby g is a locally identical transformation), therefore, at the point $r = 0$ transformation remains smooth, whence $g, f \in C^3(Q)$. Moreover, it is easy to see that $J_g(r,\varphi) = \frac{\partial}{\partial \varphi}\hat{g}(r,\varphi)$, whence $|J_g(x)| \neq 0$, $|J_f(x)| \neq 0$ for all $x \in Q$. Since $\hat{g}(r,\varphi) = g(\varphi)$ when $r > 2\varepsilon$, we see that for such r transformations g and f are rotations by variable angle depending only on polar angle φ.

---

where $|\lambda_i| = 1$ by virtue of the orthogonality. Then $\sigma(K^2) = \bigcup_{i=1}^{n} \{\lambda_i^2\}$. Since $K^2 \neq E$, $\exists \lambda_s^2 \neq 1$, i. e. $\exists \lambda_s \neq \pm 1$. That means $\operatorname{Im}\lambda_s \neq 0$, therefore, $\exists \lambda_m = \bar{\lambda}_s$ (because K has its elements real) and $\lambda_m^2 \neq 1$. Thus, there exist two eigenvalues of K^2 unequal 1, whence $r \leq n - 2$.

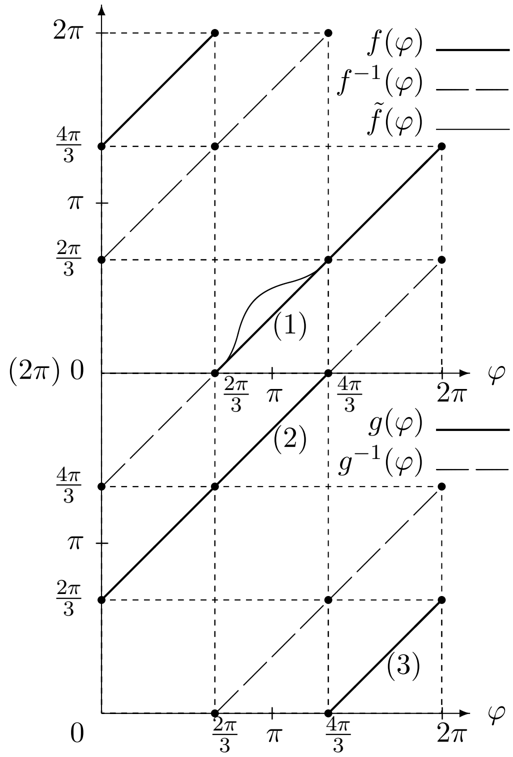

Fig. 2

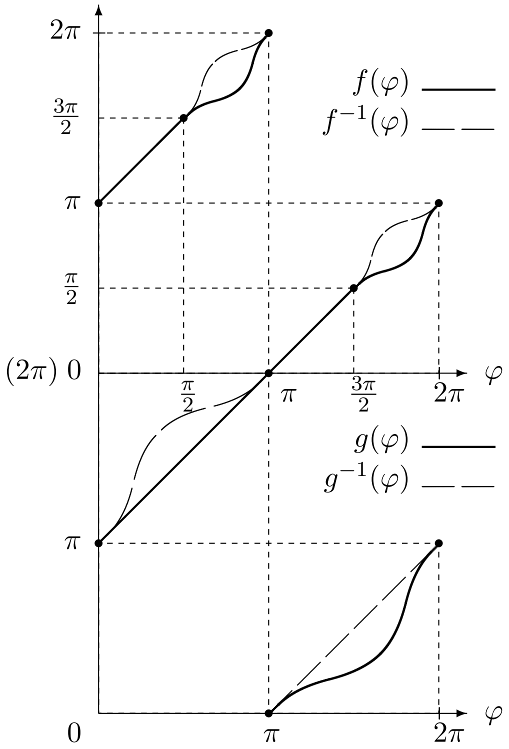

Fig. 3

Choose a function $\omega \in C^3([0,1])$ *such that* $\omega(\varphi) \geq 0$ *and* $|\omega'(\varphi)| < 1$ *for* $\varphi \in [0,1]$, $\omega(0) = 0$, $\omega(1) = 0$, *and* $\omega^{(k)}(0) = \omega^{(k)}(1) = 0$, $k = 1, 2, 3$.

Consider functions $g(\varphi) = \varphi + 2\pi/3$ *and* $f(\varphi) = \varphi + 4\pi/3$ *(see Fig. 2). Transformations* g *and* f *corresponding them satisfy properties* $(\overline{A5})$, $(\overline{A6})$, *and* $(\overline{A8})$ *for* $\varphi \in (0, 2\pi/3)$ *and* $r > 2\varepsilon$. *Set*

$$\tilde{f}(\varphi) = \begin{cases} \varphi + 4\pi/3, & \varphi \in [0, 2\pi/3] \cup [4\pi/3, 2\pi], \\ \varphi + 4\pi/3 + \frac{2\pi}{3}\,\omega\left(\frac{\varphi - 2\pi/3}{2\pi/3}\right), & \varphi \in [2\pi/3, 4\pi/3]. \end{cases}$$

It is easy to prove that $\tilde{f} \in C^3([0, 2\pi])$ *and* $\tilde{f}'(\varphi) > 0$, $\varphi \in [0, 2\pi]$. *The graph of* $\tilde{f}(\varphi)$ *is shown in Fig. 2 as the thin curve marked (1). Transformations* g *and* $\tilde{f}$ *corresponding functions* $g(\varphi)$ *and* $\tilde{f}(\varphi)$ *satisfy* $(\overline{A6})$, $(\overline{A8})$, *and* $(A5)$ *for* $\varphi \in (0, 2\pi/3)$ *and* $r > 2\varepsilon$. *Property* $(\overline{A5})$ *violates. In the same way, replacing either function* $f(\varphi)$ *on the interval* $(2\pi/3, 4\pi/3)$ *(marked (1)), function* $g(\varphi)$ *on the interval* $(2\pi/3, 4\pi/3)$ *(marked (2)), or function* $g(\varphi)$ *on the interval* $(4\pi/3, 2\pi)$ *(marked (3)) by a* C^3*-smooth strictly increasing curve (using analogous functions* $\tilde{g}(\varphi)$ *and* $\tilde{f}(\varphi)$*), we respectively violate property* $(\overline{A5})$, $(\overline{A6})$, *or* $(\overline{A8})$ *for* $\varphi \in (0, 2\pi/3)$ *and* $r > 2\varepsilon$. *It is easy to see that violating one of these properties does not affect the others. Thus, "switching* *on and off" (using or not curves (1), (2), or (3)) these properties in different combinations, we obtain all the 8 possible combinations of them.*

For any functions $g(\varphi)$ and $f(\varphi)$ considered below it is easy to prove that $g, f \in C^3([0, 2\pi])$ and $g'(\varphi) > 0,\ f'(\varphi) > 0,\ \varphi \in [0, 2\pi]$.

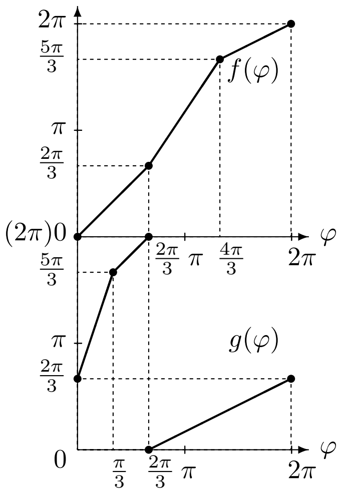

Fig. 4

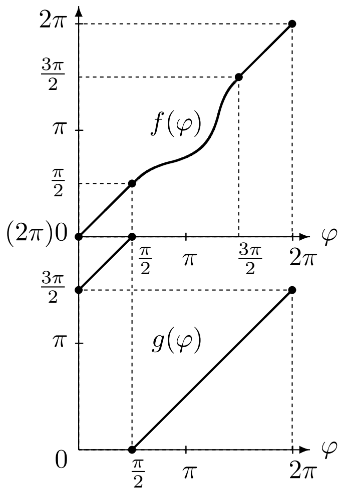

Fig. 5

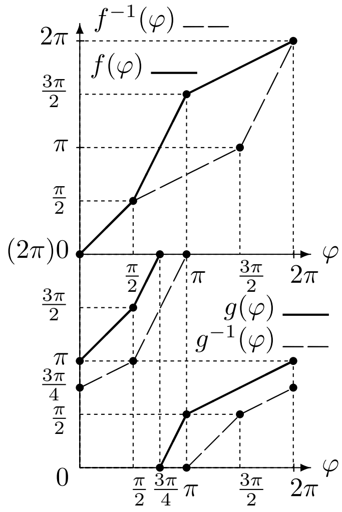

Fig. 6

Consider functions

$$g(\varphi) = \begin{cases} \varphi + \pi, & \varphi \in [0, \pi], \\ \varphi + \pi - \pi\omega\left(\frac{\varphi - \pi}{\pi}\right), & \varphi \in [\pi, 2\pi], \end{cases}$$

$$f(\varphi) = \begin{cases} \varphi + \pi, & \varphi \in [0, \pi/2] \cup [\pi, 3\pi/2], \\ \varphi + \pi - \frac{\pi}{2}\,\omega\left(\frac{\varphi - \pi/2}{\pi/2}\right), & \varphi \in [\pi/2, \pi], \\ \varphi + \pi - \frac{\pi}{2}\,\omega\left(\frac{\varphi - 3\pi/2}{\pi/2}\right), & \varphi \in [3\pi/2, 2\pi] \end{cases}$$

(see Fig. 3). Transformations g and f corresponding them satisfy properties ($\overline{A4}$) and ($\overline{A5}$) for $\varphi \in (0, \pi/2)$ and $r > 2\varepsilon$. At the same time, combining properties ($\overline{A4}$) and ($\overline{A5}$), we get $f^2(x) = x$ for $\varphi \in (0, \pi/2)$ and $r > 2\varepsilon$. However, we see that $G_2^f \neq \varnothing$.

Consider functions

$$g(\varphi) = \begin{cases} 3\varphi + \frac{2\pi}{3}, & \varphi \in [0, \pi/3], \\ \varphi + \frac{4\pi}{3}, & \varphi \in [\pi/3, 2\pi/3], \\ \frac{1}{2}\varphi + \frac{5\pi}{3}, & \varphi \in [2\pi/3, 2\pi], \end{cases} \qquad f(\varphi) = \begin{cases} \varphi, & \varphi \in [0, 2\pi/3], \\ \frac{3}{2}\varphi - \frac{\pi}{3}, & \varphi \in [2\pi/3, 4\pi/3], \\ \frac{1}{2}\varphi + \pi, & \varphi \in [4\pi/3, 2\pi] \end{cases}$$

(see Fig. 4). Transformations g and f corresponding them satisfy proper-

ties ($\overline{A6}$) *and* ($\overline{A7}$) *for* $\varphi \in (2\pi/3, 2\pi)$ *and* $r > 2\varepsilon$. *Functions*

$$g(\varphi) = \varphi + \frac{3\pi}{2}, \qquad f(\varphi) = \begin{cases} \varphi, & \varphi \in [0, \pi/2] \cup [3\pi/2, 2\pi], \\ \varphi - \pi\omega\left(\frac{\varphi - \pi/2}{\pi}\right), & \varphi \in [\pi/2, 3\pi/2] \end{cases}$$

(see Fig. 5) produce transformations g *and* f *with properties* ($\overline{A3}$) *and* ($\overline{A7}$) *for* $\varphi \in (0, \pi/2)$ *and* $r > 2\varepsilon$. *Functions*

$$g(\varphi) = \begin{cases} \varphi + \pi, & \varphi \in [0, \pi/2], \\ 2\varphi + \frac{\pi}{2}, & \varphi \in [\pi/2, \pi], \\ \frac{1}{2}\varphi + 2\pi, & \varphi \in [\pi, 2\pi], \end{cases} \qquad f(\varphi) = \begin{cases} \varphi, & \varphi \in [0, \pi/2], \\ 2\varphi - \frac{\pi}{2}, & \varphi \in [\pi/2, \pi], \\ \frac{1}{2}\varphi + \pi, & \varphi \in [\pi, 2\pi] \end{cases}$$

(see Fig. 6) produce transformations g *and* f *with properties* ($\overline{A3}$) *and* ($\overline{A8}$) *for* $\varphi \in (0, \pi/2)$ *and* $r > 2\varepsilon$.

Next we prove a statement stronger than Lemma 3.

Lemma 4. *Suppose that* $G_g^2 \neq \varnothing$, $G_f^2 \neq \varnothing$, *and* $g(Q) = f(Q) = Q$. *Let operator* A *be normal and conditions* $a_1 - a_2 \neq 0$ *and* $a_1 + a_2 \neq 0$ *hold. Then*

$$g(x) = Kx + b, \quad f(x) = Cx + d, \quad x \in Q.$$

Here K *and* C *are orthogonal matrices of* $n \times n$ *size such that* $K^2 \neq E$ *and* $C^2 \neq E$, $b, d \in \mathbb{R}^n$.

Proof. This Lemma states the same result as Lemma 3 while the assumptions are the same as in Lemma 3 except for condition $a_1 a_2 < 0$ which is changed to condition $a_1 - a_2 \neq 0$. Therefore, the proof totally coincides with the proof of Lemma 3 except for part **4**. This (and only this) part of the proof of Lemma 3 uses condition $a_1 a_2 < 0$. Now we shall obtain the same result, using condition $a_1 - a_2 \neq 0$.

1. Let assumptions from part **4** of the proof of Lemma 3 be fulfilled, i. e. in a neighborhood of a point $x^0 \in G_g^2$ property ($\overline{A5}$) holds. In the same way as in part **4** of the proof of Lemma 3 we obtain Eq. (4.15). Let us prove that $a_1 - a_2 |J_g(x)| \neq 0$ if $a_1 - a_2 \neq 0$.

Since $g(Q) = Q$, $g \in C^3(Q)$, and $|J_g(x)| \neq 0$ for any $x \in Q$, transformation $g : Q \mapsto Q$ is a one-to-one mapping and $g^{-1}(Q) = Q$. Denote the measure of the set Q by $|Q|$. We can write:

$$|Q| = \int_Q dx = \int_{g^{-1}(Q)} d(g(y)) = \int_Q |J_g(y)| dy, \quad \text{i. e.,} \quad \int_Q dx = \int_Q |J_g(x)| dx.$$

This implies that if $|J_g(x)| = \text{const}$ for $x \in Q$, then $|J_g(x)| = 1$. Since $|J_g(x)| \in C^2(Q)$, if $|J_g(x^0)| > 1$ (or $|J_g(x^0)| < 1$) holds in some set, then this set has a non-null measure. Therefore, if $|J_g(x^0)| > 1$ (or $|J_g(x^0)| < 1$) holds in some set, then there exist a non-null set where $|J_g(x^0)| < 1$ (or $|J_g(x^0)| > 1$) and a non-null set where $|J_g(x)|$ changes C^2-smoothly between these values.

Suppose that $a_1 - a_2|J_g(x^0)| = 0$ holds at some $x^0 \in G_g^2$. Parts **2**–**10** of the proof of Lemma 3 imply that if $x \in G_g^2$, then it is either $|J_g(x)| = 1$ or $|J_g(x)| = a_1/a_2$. Under conditions of the lemma $a_1 - a_2 \neq 0$ whence $a_1/a_2 \neq 1$. That means there exists a set where $|J_g(x)|$ changes C^2-smoothly between values a_1/a_2 and 1. But this set cannot intersect G_g^2. Therefore, $|J_g(x)| = a_1/a_2$ for all $x \in \bar{G}_g^{2,0}$, where $\bar{G}_g^{2,0}$ is a closed connected component of $\overline{G_g^2}$ such that $x^0 \in \bar{G}_g^{2,0}$. But equality $|J_g(x)| = a_1/a_2$ is possible only when property $(\overline{A5})$ holds, therefore, this property holds for all $x \in \bar{G}_g^{2,0}$. By H denote the connected component of the set $\{x \in Q : g(x) = f^{-1}(x)\}$ such that $x^0 \in H$. It is obvious that H is a closed set. Then $\bar{G}_g^{2,0} \subseteq H$.

Let us consider where $|J_g(x)|$ can change C^2-smoothly between a_1/a_2 and 1.

2. Suppose that set $H \setminus \bar{G}_g^{2,0}$ also has a non-empty interior. Then set $(H \setminus \bar{G}_g^{2,0}) \cap \widetilde{G}_g^2$ has a non-empty interior. Choose a point x^1 such that $B_{2\varepsilon}(x^1) \subset (H \setminus \bar{G}_g^{2,0}) \cap \widetilde{G}_g^2$. If set $(H \setminus \bar{G}_g^{2,0}) \cap \widetilde{G}_g^2$ contains points where $|J_g(x)| \neq 1$, this is possible only when $g(x) \neq x$, therefore, choose $x^1 \neq g(x^1)$ and $B_{2\varepsilon}(x^1)$ such that $g(B_{2\varepsilon}(x^1)) \cap B_{2\varepsilon}(x^1) = \varnothing$. For $x \in B_{2\varepsilon}(x^1)$ we have $g(x) = g^{-1}(x) = f^{-1}(x)$. Therefore, $f(x) \neq x$ should hold (otherwise $f^{-1}(x) = x = g(x)$), so choose $B_{2\varepsilon}(x^1)$ such that $f(B_{2\varepsilon}(x^1)) \cap B_{2\varepsilon}(x^1) = \varnothing$. Introduce a cut-off function $\eta \in \dot{C}^\infty(Q)$ such that $0 \leq \eta(x) \leq 1$ for $x \in Q$, $\eta(x) = 1$ for $x \in B_\varepsilon(x^1)$, and $\operatorname{supp}\eta \subset B_{2\varepsilon}(x^1)$. By the normality of A we have $AA^*\eta = A^*A\eta$. By definitions of $B_{2\varepsilon}(x^1)$ and $\eta(x)$, taking into account that $g(x) = g^{-1}(x) = f^{-1}(x)$, for $x \in B_\varepsilon(x^1)$ we get:

$$\begin{aligned}
&A_0A_1\eta(x) = \Delta[a_1\eta(g(x))] = 0, && A_1A_0\eta(x) = a_1(\Delta\eta)(g(x)) = 0,\\
&A_0A_2\eta(x) = \Delta[a_2\eta(f(x))] = 0, && A_2A_0\eta(x) = a_2(\Delta\eta)(f(x)) = 0,\\
&A_1A_1^*\eta(x) = a_1^2|J_{g^{-1}}(g(x))|, && A_2A_2^*\eta(x) = a_2^2|J_{f^{-1}}(f(x))|,\\
&A_1^*A_1\eta(x) = a_1^2|J_{g^{-1}}(x)|, && A_2^*A_2\eta(x) = a_2^2|J_{f^{-1}}(x)| = a_2^2|J_{g^{-1}}(x)|,\\
&A_0A_1^*\eta(x) = \Delta[a_1|J_{g^{-1}}(x)|\eta(g^{-1}(x))] = 0,\\
&A_0A_2^*\eta(x) = \Delta[a_2|J_{f^{-1}}(x)|\eta(f^{-1}(x))] = 0,\\
&A_1^*A_0\eta(x) = a_1|J_{g^{-1}}(x)|(\Delta\eta)(g^{-1}(x)) = 0,
\end{aligned}$$

$$A_2^*A_0\eta(x) = a_2|J_{f^{-1}}(x)|(\Delta\eta)(f^{-1}(x)) = 0,$$
$$A_1^*A_2\eta(x) = a_1a_2|J_{g^{-1}}(x)|\eta(fg^{-1}(x)) = a_1a_2|J_{g^{-1}}(x)|,$$
$$A_2^*A_1\eta(x) = a_1a_2|J_{f^{-1}}(x)|\eta(gf^{-1}(x)) = a_1a_2|J_{f^{-1}}(x)| = a_1a_2|J_{g^{-1}}(x)|.$$

a). Suppose that $g(x) \neq f(x)$ for $x \in B_{2\varepsilon}(x^1)$. Then

$$A_1A_2^*\eta(x) = a_1a_2|J_{f^{-1}}(g(x))|\eta(f^{-1}g(x)) = 0,$$
$$A_2A_1^*\eta(x) = a_1a_2|J_{g^{-1}}(f(x))|\eta(g^{-1}f(x)) = 0.$$

By the normality of A this implies for $x \in B_\varepsilon(x^1)$

$$a_1^2|J_{g^{-1}}(g(x))| + a_2^2|J_{f^{-1}}(f(x))| = a_1^2|J_{g^{-1}}(x)| + a_2^2|J_{f^{-1}}(x)| + a_1a_2|J_{g^{-1}}(x)| + a_1a_2|J_{f^{-1}}(x)|,$$
$$a_1^2|J_g(x)|^{-1} + a_2^2|J_f(x)|^{-1} = (a_1+a_2)^2|J_f(x)|.$$

As $f(x) \neq f^{-1}(x)$ for $x \in B_{2\varepsilon}(x^1)$, we get $B_{2\varepsilon}(x^1) \subset G_f^2$. Since $f(x) \neq g^{-1}(x)$ for $x \in B_{2\varepsilon}(x^1)$, we obtain $|J_f(x)| = 1$ for $x \in B_{2\varepsilon}(x^1)$ (this follows from parts **2**–**10** of the proof of Lemma 3). Therefore, $|J_g(x)| = \text{const}$ for $x \in B_\varepsilon(x^1)$.

b). Suppose that $g(x) = f(x)$ for $x \in B_{2\varepsilon}(x^1)$. Then

$$A_1A_2^*\eta(x) = a_1a_2|J_{f^{-1}}(g(x))|, \qquad A_2A_1^*\eta(x) = a_1a_2|J_{g^{-1}}(f(x))|.$$

By the normality of A this implies for $x \in B_\varepsilon(x^1)$

$$a_1^2|J_{g^{-1}}(g(x))| + a_2^2|J_{f^{-1}}(f(x))| + a_1a_2|J_{f^{-1}}(g(x))| + a_1a_2|J_{g^{-1}}(f(x))| = a_1^2|J_{g^{-1}}(x)| + a_2^2|J_{f^{-1}}(x)| + a_1a_2|J_{g^{-1}}(x)| + a_1a_2|J_{f^{-1}}(x)|,$$
$$(a_1+a_2)^2|J_g(x)|^{-1} = (a_1+a_2)^2|J_g(x)|$$

whence $|J_g(x)| = 1$ for $x \in B_\varepsilon(x^1)$.

Thus, $|J_g(x)| = \text{const}$ for $x \in (H \setminus \bar{G}_g^{2,0}) \cap \widetilde{G}_g^2$. It is obvious that the boundary of set $(H \setminus \bar{G}_g^{2,0}) \cap \widetilde{G}_g^2$ partially coincides with boundary $\partial\bar{G}_g^{2,0}$. Therefore, value $|J_g(x)| = a_1/a_2$ is extended from the set $\bar{G}_g^{2,0}$, i. e., $|J_g(x)| = a_1/a_2$ for $x \in (H \setminus \bar{G}_g^{2,0}) \cap \widetilde{G}_g^2$. By the assumptions of lemma $a_1 \neq a_2$, so case **2.b** is not realized. Therefore, case **2.a** take place, moreover, it is easy to prove that $|J_g(x)| = a_1/a_2$ only under condition $a_1 = a_2(-1 \pm \sqrt{5})/2$. Therefore, the set $(H \setminus \bar{G}_g^{2,0}) \cap \widetilde{G}_g^2$ either has empty interior (as well as $H \setminus \bar{G}_g^{2,0}$), or it has a non-empty interior, $|J_g(x)| = a_1/a_2$ on this set, and $a_1 = a_2(-1 \pm \sqrt{5})/2$ holds.

Let the set $(H \setminus \bar{G}_g^{2,0}) \cap G_g^2$ have a non-empty interior. Then value $|J_g(x)| = a_1/a_2$ extends to this set because $|J_g(x)|$ is smooth. Thus, $|J_g(x)| = a_1/a_2$ for all $x \in H$. At the same time, $\partial H \subset \widetilde{G}_g^2$. Indeed, otherwise there should exist a subset of the set $G_g^2 \setminus H$ such that value $|J_g(x)| = a_1/a_2$ extends from H to this subset while property $(\overline{A5})$ does not hold on this subset, which is impossible. Moreover, there exists an outer neighborhood of boundary of the set H belonging to the set $\widetilde{G}_g^2$. Indeed, otherwise the set $G_g^2 \setminus H$ has a part of its boundary common with the set H, but $|J_g(x)| = 1$ for $x \in G_g^2 \setminus H$, which contradicts the smoothness of $|J_g(x)|$.

The same reasoning is valid when the set $(H \setminus \bar{G}_g^{2,0}) \cap G_g^2$ has empty interior or the set $(H \setminus \bar{G}_g^{2,0}) \cap \widetilde{G}_g^2$ has empty interior ($H \setminus \bar{G}_g^{2,0}$ has empty interior). Thus, at any case $|J_g(x)| = a_1/a_2$ for $x \in H$ and there exist an outer neighborhood of the set H belonging to the set $\widetilde{G}_g^2$. Denote this neighborhood by $O_+(\partial H)$.

3. Choose a point $x^1 \in O_+(\partial H)$ such that $B_{2\varepsilon}(x^1) \subset O_+(\partial H)$. Since $O_+(\partial H) \subset \widetilde{G}_g^2$, we get $g(x) = g^{-1}(x)$ for $x \in O_+(\partial H)$. As $B_{2\varepsilon}(x^1) \subset O_+(\partial H)$, we have $g(x) \neq f^{-1}(x)$ for $x \in B_{2\varepsilon}(x^1)$. If set $O_+(\partial H)$ contains points where $|J_g(x)| \neq 1$, this is possible only when $g(x) \neq x$, therefore, choose $x^1 \neq g(x^1)$ and $B_{2\varepsilon}(x^1)$ such that $g(B_{2\varepsilon}(x^1)) \cap B_{2\varepsilon}(x^1) = \varnothing$. Introduce a cut-off function $\eta \in \dot{C}^\infty(Q)$ such that $0 \leq \eta(x) \leq 1$ for $x \in Q$, $\eta(x) = 1$ for $x \in B_\varepsilon(x^1)$, and $\operatorname{supp} \eta \subset B_{2\varepsilon}(x^1)$.

a). Suppose that $f(x) = x$ for $x \in B_{2\varepsilon}(x^1)$. Then for $x \in B_\varepsilon(x^1)$ we get:

$$
\begin{aligned}
&A_0A_1\eta(x) = \Delta[a_1\eta(g(x))] = 0, \quad A_0A_1^*\eta(x) = \Delta[a_1|J_{g^{-1}}(x)|\eta(g^{-1}(x))] = 0,\\
&A_1A_0\eta(x) = a_1(\Delta\eta)(g(x)) = 0, \quad A_1^*A_0\eta(x) = a_1|J_{g^{-1}}(x)|(\Delta\eta)(g^{-1}(x)) = 0,\\
&A_1A_1^*\eta(x) = a_1^2|J_{g^{-1}}(g(x))|, \qquad A_2A_2^*\eta(x) = a_2^2|J_{f^{-1}}(f(x))| = a_2^2,\\
&A_1^*A_1\eta(x) = a_1^2|J_{g^{-1}}(x)|, \qquad A_2^*A_2\eta(x) = a_2^2|J_{f^{-1}}(x)| = a_2^2,\\
&A_0A_2\eta(x) = \Delta[a_2\eta(f(x))] = \Delta[a_2\eta(x)] = 0,\\
&A_0A_2^*\eta(x) = \Delta[a_2|J_{f^{-1}}(x)|\eta(f^{-1}(x))] = \Delta[a_2\eta(x)] = 0,\\
&A_2A_0\eta(x) = a_2(\Delta\eta)(f(x)) = a_2\Delta\eta(x) = 0,\\
&A_2^*A_0\eta(x) = a_2|J_{f^{-1}}(x)|(\Delta\eta)(f^{-1}(x)) = a_2\Delta\eta(x) = 0,\\
&A_1A_2^*\eta(x) = a_1a_2|J_{f^{-1}}(g(x))|\eta(f^{-1}g(x)) = 0,\\
&A_2A_1^*\eta(x) = a_1a_2|J_{g^{-1}}(f(x))|\eta(g^{-1}f(x)) = 0,\\
&A_1^*A_2\eta(x) = a_1a_2|J_{g^{-1}}(x)|\eta(fg^{-1}(x)) = 0,\\
&A_2^*A_1\eta(x) = a_1a_2|J_{f^{-1}}(x)|\eta(gf^{-1}(x)) = 0.
\end{aligned}
$$

By the normality of A we have $AA^*\eta = A^*A\eta$ whence for $x \in B_\varepsilon(x^1)$ we obtain

$$a_1^2|J_{g^{-1}}(g(x))| + a_2^2 = a_1^2|J_{g^{-1}}(x)| + a_2^2.$$

Taking into account that $g(x) = g^{-1}(x)$ for $x \in B_\varepsilon(x^1)$, we get $|J_g(x)| = 1$ for $x \in B_\varepsilon(x^1)$.

b). Suppose that $f(x) \neq x$ for $x \in B_{2\varepsilon}(x^1)$. Then choose ε such that $f(B_{2\varepsilon}(x^1)) \cap B_{2\varepsilon}(x^1) = \varnothing$. Let relations $f(x) \neq g(x)$ and $f(x) \neq f^{-1}(x)$ hold. Then $f(x) \neq g(x) = g^{-1}(x) \neq f^{-1}(x)$ and $f(x) \neq f^{-1}(x)$ for $x \in B_{2\varepsilon}(x^1)$. Choose ε such that additionally $f^{-1}(B_{2\varepsilon}(x^1)) \cap B_{2\varepsilon}(x^1) = \varnothing$, $f(B_{2\varepsilon}(x^1)) \cap g(B_{2\varepsilon}(x^1)) = \varnothing$, and $f^{-1}(B_{2\varepsilon}(x^1)) \cap g^{-1}(B_{2\varepsilon}(x^1)) = \varnothing$. Then for $x \in B_\varepsilon(x^1)$ we get:

$$\begin{aligned}
&A_0A_1\eta(x) = \Delta[a_1\eta(g(x))] = 0, && A_1A_0\eta(x) = a_1(\Delta\eta)(g(x)) = 0,\\
&A_0A_2\eta(x) = \Delta[a_2\eta(f(x))] = 0, && A_2A_0\eta(x) = a_2(\Delta\eta)(f(x)) = 0,\\
&A_1A_1^*\eta(x) = a_1^2|J_{g^{-1}}(g(x))|, && A_2A_2^*\eta(x) = a_2^2|J_{f^{-1}}(f(x))|,\\
&A_1^*A_1\eta(x) = a_1^2|J_{g^{-1}}(x)|, && A_2^*A_2\eta(x) = a_2^2|J_{f^{-1}}(x)|,\\
&A_0A_1^*\eta(x) = \Delta[a_1|J_{g^{-1}}(x)|\eta(g^{-1}(x))] = 0,\\
&A_0A_2^*\eta(x) = \Delta[a_2|J_{f^{-1}}(x)|\eta(f^{-1}(x))] = 0,\\
&A_1^*A_0\eta(x) = a_1|J_{g^{-1}}(x)|(\Delta\eta)(g^{-1}(x)) = 0,\\
&A_2^*A_0\eta(x) = a_2|J_{f^{-1}}(x)|(\Delta\eta)(f^{-1}(x)) = 0,\\
&A_1A_2^*\eta(x) = a_1a_2|J_{f^{-1}}(g(x))|\eta(f^{-1}g(x)) = 0,\\
&A_2A_1^*\eta(x) = a_1a_2|J_{g^{-1}}(f(x))|\eta(g^{-1}f(x)) = 0,\\
&A_1^*A_2\eta(x) = a_1a_2|J_{g^{-1}}(x)|\eta(fg^{-1}(x)) = 0,\\
&A_2^*A_1\eta(x) = a_1a_2|J_{f^{-1}}(x)|\eta(gf^{-1}(x)) = 0.
\end{aligned}$$

By the normality of A we have $AA^*\eta = A^*A\eta$ whence for $x \in B_\varepsilon(x^1)$ we obtain

$$a_1^2|J_{g^{-1}}(g(x))| + a_2^2|J_{f^{-1}}(f(x))| = a_1^2|J_{g^{-1}}(x)| + a_2^2|J_{f^{-1}}(x)|,$$
$$a_1^2(|J_g(x)|^{-1} - |J_g(x)|) = a_2^2(|J_{f^{-1}}(x)| - |J_f(x)|^{-1}).$$

As $f(x) \neq f^{-1}(x)$ for $x \in B_{2\varepsilon}(x^1)$, we get $B_{2\varepsilon}(x^1) \subset G_f^2$; moreover, $f(x) \neq g^{-1}(x)$, therefore, $|J_f(x)| = 1$ for $x \in B_{2\varepsilon}(x^1)$ (this follows from parts **2**–**10** of the proof of Lemma 3). Also $f^{-1}(B_{2\varepsilon}(x^1)) \subset G_f^2$. We claim that $f(y) \neq g^{-1}(y)$ for $y \in f^{-1}(B_{2\varepsilon}(x^1))$. Assume the contrary: suppose that there exists a point $y^* \in f^{-1}(B_{2\varepsilon}(x^1))$ such that $f(y) = g^{-1}(y)$. Then there exists a point $x^* \in B_{2\varepsilon}(x^1)$ such that $f^{-1}(x^*) = y^*$. Hence $f(f^{-1}(x^*)) = g^{-1}(f^{-1}(x^*))$, thereby $x^* = g^{-1}(f^{-1}(x^*))$ and $g(x^*) = f^{-1}(x^*)$. This contradicts condition $g(x) \neq f^{-1}(x)$ for $x \in B_{2\varepsilon}(x^1)$, therefore, $f(y) \neq g^{-1}(y)$ holds for $y \in f^{-1}(B_{2\varepsilon}(x^1))$. Since $f^{-1}(B_{2\varepsilon}(x^1)) \subset G_f^2$ and $f(y) \neq g^{-1}(y)$ for $y \in f^{-1}(B_{2\varepsilon}(x^1))$, we get $|J_{f^{-1}}(x)| = |J_f(f^{-1}(x))| = 1$ for $x \in B_{2\varepsilon}(x^1)$. Taking this into account, we obtain $a_1^2(|J_g(x)|^{-1} - |J_g(x)|) = 0$ whence $|J_g(x)| = 1$ for $x \in B_{2\varepsilon}(x^1)$.

c). Suppose that $f(x) \neq x$ and $f(x) = g(x)$, i. e., $f(x) = g(x) = g^{-1}(x) \neq f^{-1}(x)$ for $x \in B_{2\varepsilon}(x^1)$. Choose ε such that $f(B_{2\varepsilon}(x^1)) \cap B_{2\varepsilon}(x^1) = \varnothing$, $f^{-1}(B_{2\varepsilon}(x^1)) \cap B_{2\varepsilon}(x^1) = \varnothing$, and $f^{-1}(B_{2\varepsilon}(x^1)) \cap g^{-1}(B_{2\varepsilon}(x^1)) = \varnothing$. Then for $x \in B_\varepsilon(x^1)$ we get:

$$\begin{aligned}
&A_0A_1\eta(x) = \Delta[a_1\eta(g(x))] = 0, \qquad A_1A_0\eta(x) = a_1(\Delta\eta)(g(x)) = 0,\\
&A_0A_2\eta(x) = \Delta[a_2\eta(f(x))] = 0, \qquad A_2A_0\eta(x) = a_2(\Delta\eta)(f(x)) = 0,\\
&A_1A_1^*\eta(x) = a_1^2|J_{g^{-1}}(g(x))|, \qquad A_2A_2^*\eta(x) = a_2^2|J_{f^{-1}}(f(x))|,\\
&A_1^*A_1\eta(x) = a_1^2|J_{g^{-1}}(x)|, \qquad A_2^*A_2\eta(x) = a_2^2|J_{f^{-1}}(x)|,\\
&A_0A_1^*\eta(x) = \Delta[a_1|J_{g^{-1}}(x)|\eta(g^{-1}(x))] = 0,\\
&A_0A_2^*\eta(x) = \Delta[a_2|J_{f^{-1}}(x)|\eta(f^{-1}(x))] = 0,\\
&A_1^*A_0\eta(x) = a_1|J_{g^{-1}}(x)|(\Delta\eta)(g^{-1}(x)) = 0,\\
&A_2^*A_0\eta(x) = a_2|J_{f^{-1}}(x)|(\Delta\eta)(f^{-1}(x)) = 0,\\
&A_1A_2^*\eta(x) = a_1a_2|J_{f^{-1}}(g(x))|\eta(f^{-1}g(x)) = a_1a_2|J_{f^{-1}}(g(x))|,\\
&A_2A_1^*\eta(x) = a_1a_2|J_{g^{-1}}(f(x))|\eta(g^{-1}f(x)) = a_1a_2|J_{g^{-1}}(f(x))|,\\
&A_1^*A_2\eta(x) = a_1a_2|J_{g^{-1}}(x)|\eta(fg^{-1}(x)) = 0,\\
&A_2^*A_1\eta(x) = a_1a_2|J_{f^{-1}}(x)|\eta(gf^{-1}(x)) = 0.
\end{aligned}$$

By the normality of A we have $AA^*\eta = A^*A\eta$ whence for $x \in B_\varepsilon(x^1)$ we obtain

$$\begin{aligned}
&a_1^2|J_{g^{-1}}(g(x))| + a_2^2|J_{f^{-1}}(f(x))| + a_1a_2|J_{f^{-1}}(g(x))| + a_1a_2|J_{g^{-1}}(f(x))|\\
&\qquad\qquad = a_1^2|J_{g^{-1}}(x)| + a_2^2|J_{f^{-1}}(x)|,\\
&(a_1+a_2)^2|J_g(x)|^{-1} - a_1^2|J_g(x)| = a_2^2|J_{f^{-1}}(x)|.
\end{aligned}$$

In the same way as in the case **b)**, we get $|J_{f^{-1}}(x)| = 1$ whence $|J_g(x)| = \text{const}$ for $x \in B_\varepsilon(x^1)$.

d). Suppose that $f(x) \neq x$ and $f(x) = f^{-1}(x)$, i. e., $g(x) = g^{-1}(x) \neq f(x) = f^{-1}(x)$ for $x \in B_{2\varepsilon}(x^1)$. Choose ε such that $f(B_{2\varepsilon}(x^1)) \cap B_{2\varepsilon}(x^1) = \varnothing$, $f^{-1}(B_{2\varepsilon}(x^1)) \cap B_{2\varepsilon}(x^1) = \varnothing$, $f(B_{2\varepsilon}(x^1)) \cap g(B_{2\varepsilon}(x^1)) = \varnothing$, and $f^{-1}(B_{2\varepsilon}(x^1)) \cap$

$g^{-1}(B_{2\varepsilon}(x^1)) = \varnothing$. Then by the normality of A we get the same relations as in the case **b)** whence for $x \in B_\varepsilon(x^1)$ we obtain

$$a_1^2(|J_g(x)|^{-1} - |J_g(x)|) = a_2^2(|J_f(x)| - |J_f(x)|^{-1}). \tag{4.24}$$

We see that jacobians $|J_g(x)|$ and $|J_f(x)|$ can change simultaneously while this relation holds.

Denote the subsets of Q corresponding cases **a)**, **b)**, **c)**, and **d)** by $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ respectively. It was proved that in sets $\mathcal{A}$ and $\mathcal{B}$ we have $|J_g(x)| = 1$. Therefore, since $|J_g(x)|$ is smooth, these sets cannot have boundaries coinciding with the boundary of set H because $|J_g(x)| = a_1/a_2$ when $x \in H$. It is easy to prove that in set $\mathcal{C}$ we get $|J_g(x)| = 1$ only if $a_1a_2 = 0$ and we get $|J_g(x)| = a_1/a_2$ only if either a_1/a_2 equals -1 or the irrational number $1,47...$. In the last case set $\mathcal{C}$ can have its boundary coinciding with the boundary of set H.

Therefore, set $\mathcal{D}$ has a part Γ of its boundary such that $|J_g(x)| = a_1/a_2$ for $x \in \Gamma$. This part of boundary can coincide either with ∂H or $\partial\mathcal{C}$. Suppose that $z \in \Gamma$. Then proceeding Eq. (4.24) to limit as $x \to z$, we get $|J_f(x)| = (q + \sqrt{q^2+4})/2$, where $q = a_1/a_2 - (a_1/a_2)^3$. This implies that $|J_f(x)| = 1$ or $|J_f(x)| = a_2/a_1$ only if $|a_1| = |a_2|$ which is impossible under conditions of the lemma. Therefore, $\Gamma \subset \widetilde{G}_f^2$. As $|J_f(x)|$ is smooth, there exists a neighborhood $O(\Gamma)$ of Γ such that $O(\Gamma) \subset \widetilde{G}_f^2$. But $\mathcal{C} \cap \widetilde{G}_f^2 = \varnothing$, therefore, set $\mathcal{C}$ cannot have a part of its boundary where $|J_g(x)| = a_1/a_2$ coinciding with the boundary of set $\mathcal{D}$. Therefore, $\Gamma \subset \partial H$. Denote $O_-(\Gamma) = O(\Gamma) \cap H$.

Consider a set $O_-(\Gamma) \cap \widetilde{G}_g^2$. We get $g(x) = g^{-1}(x) = f(x) = f^{-1}(x)$ for $x \in O_-(\Gamma) \cap \widetilde{G}_g^2$. In part **2.b** it was proved that $|J_f(x)| = 1$ in this case. Hence this set cannot approach closely to Γ, otherwise the smoothness of $|J_f(x)|$ would be violated. Therefore, the set $O_-(\Gamma) \cap\ G_g^2$ with a non-empty interior approaches closely to Γ. On the other hand, in part **10** of the proof of Lemma 3 it was proved that in this case the normality of A implies $a_1 = 0$ which contradicts the initial assumptions.

Thus, C^2-smooth changing of $|J_g(x)|$ between a_1/a_2 and 1 leads to the contradiction with the initial assumptions. Therefore, $|J_g(x)| = 1$ for $x \in G_g^2$ which proves part **4** of the proof of Lemma 3 under conditions of the present lemma. □

LEMMA 5. *Suppose that $G_g^2 \neq \varnothing$, $G_f^2 \neq \varnothing$, and $g(Q) = f(Q) = Q$. Let operator A be normal and condition $|a_1| \neq |a_2|$ hold. Then $gf(x) = fg(x)$ for all $x \in Q$.*

Proof. It follows from Lemma 4 that transformations g and f has form (2.1).

By definition, $D(A) = \{u \in W_2^2(Q) : Bu = 0\}$. Therefore, by condition $g(Q) = f(Q) = Q$ and form (2.1) of transformations g and f we get $A_1u, A_2u, A_1^*u, A_2^*u \in D(A)$ if $u \in D(A)$. Then by the theorem on smoothness of generalized solutions of elliptic equations near the boundary ([9], Chap. 2, Sec. 5, Theorem 5.1) we obtain $D(AA^*) = D(A^*A) = \{u \in W_2^4(Q) : Bu = B\Delta u = 0\}$.

By the normality of operator A we get

$$(A_0+A_1+A_2)(A_0+A_1^*+A_2^*)u = (A_0+A_1^*+A_2^*)(A_0+A_1+A_2)u, \quad u \in D(AA^*). \tag{4.25}$$

Form (2.1) of transformations g and f implies that equations (4.3)–(4.4) hold identically in Q. Combining this with Eq. (4.2) and considering the analogous equation for f, we get

$$A_0A_1u = A_1A_0u, \quad A_0A_2u = A_2A_0u, \quad u \in D(AA^*). \tag{4.26}$$

On the other hand, $g^{-1}(y) = K^{-1}y - K^{-1}b$. Since matrix K^{-1} also is orthogonal, from equations (4.3)–(4.4) and identity $|J_{g^{-1}}(x)| \equiv 1$ we get (using the same reasoning for f):

$$A_0A_1^*u = A_1^*A_0u, \quad A_0A_2^*u = A_2^*A_0u, \quad u \in D(AA^*). \tag{4.27}$$

Taking into account equations (4.26) and (4.27), from Eq. (4.25) we get

$$(A_1 + A_2)(A_1^* + A_2^*)u = (A_1^* + A_2^*)(A_1 + A_2)u, \quad u \in D(AA^*).$$

Using Lemma 1 and identity $|J_g(x)| = |J_f(x)| = 1$ ($x \in Q$), for all $u \in D(AA^*)$ we obtain

$$u(f^{-1}g(x)) + u(g^{-1}f(x)) = u(fg^{-1}(x)) + u(gf^{-1}(x)). \tag{4.28}$$

This implies that a couple of points $\{f^{-1}g(x),\, g^{-1}f(x)\}$ coincides with a couple $\{fg^{-1}(x),\, gf^{-1}(x)\}$ for all $x \in Q$. Indeed, assume the contrary: suppose that there exists a point $x_0 \in Q$ such that couples of points $\{f^{-1}g(x_0),\, g^{-1}f(x_0)\}$ and $\{fg^{-1}(x_0),\, gf^{-1}(x_0)\}$ are not the same. Then a set $\mathcal{K}(x^0) = \{f^{-1}g(x_0),\, g^{-1}f(x_0)\} \setminus \{fg^{-1}(x_0),\, gf^{-1}(x_0)\}$ is not empty (clearly, it consists of one or two points). Since transformations g and f are smooth, there exists a neighborhood of the point x^0 where inequalities between points $f^{-1}g(x)$, $g^{-1}f(x)$, $fg^{-1}(x)$, and $gf^{-1}(x)$ remain valid (however, equalities can became violated). Denote $y_1(x) = f^{-1}g(x)$ and $y_2(x) = g^{-1}f(x)$, then

$\mathcal{K}(x^0) = \bigcup_{i=p}^{q} \{y_i(x^0)\}$, where $1 \le p \le q \le 2$. Choose any integer k such that $p \le k \le q$. It is possible to choose a ball $B_{2\delta}(x^0)$ such that $y_k(B_{2\delta}(x^0)) \cap fg^{-1}(B_{2\delta}(x^0)) = \varnothing$ and $y_k(B_{2\delta}(x^0)) \cap gf^{-1}(B_{2\delta}(x^0)) = \varnothing$. Introduce a function $\xi \in \dot{C}^\infty(\mathbb{R}^n)$ such that $0 \le \xi(x) \le 1$ for any $x \in \mathbb{R}^n$, $\xi(x) = 1$ for all $x \in y_k(B_\delta(x^0))$, and $\operatorname{supp} \xi \subset y_k(B_{2\delta}(x^0))$. Putting $u = \xi(x)$, we get $u(fg^{-1}(x)) = 0$, $u(gf^{-1}(x)) = 0$, and $u(y_k(x)) = 1$ for $x \in B_\delta(x^0)$. Therefore, for $x \in B_\delta(x^0)$ we get zero on the right-hand side of Eq. (4.28) while the left-hand side is not less than one. This contradiction proves that considered couples of points coincide for all $x \in Q$.

Therefore, for any $x \in Q$ at least one couple of equalities holds:

$$f^{-1}g(x) = fg^{-1}(x), \qquad g^{-1}f(x) = gf^{-1}(x), \tag{4.29}$$

$$f^{-1}g(x) = gf^{-1}(x), \qquad g^{-1}f(x) = fg^{-1}(x). \tag{4.30}$$

Since g and f are one-to-one transformations, equalities (4.29) imply $g^2(x) = f^2(x)$ and equalities (4.30) imply $gf(x) = fg(x)$.

At least one of equalities $gf(x) = fg(x)$ and $g^2(x) = f^2(x)$ holds for all $x \in Q$. Indeed, since g and f have form (2.1), either of equalities considered is a linear system of equations. A solution of such a system is a hyperplane of dimension $n-r$, where r is a rank of a system matrix. Clearly, if any point $x \in Q$ is a solution of at least one of these systems, then at least one system has a zero rank matrix, therefore, at least one equality holds identically in Q.

We claim that identity $g^2(x) = f^2(x)$ ($x \in Q$) implies $gf(x) = fg(x)$ ($x \in Q$). Indeed, since g and f have form (2.1), identity $g^2(x) = f^2(x)$ ($x \in Q$) implies $K^2 = C^2$. As matrix K^2 is orthogonal, it can be expressed in a form $K^2 = S^{-1}US$, where S is some orthogonal matrix, $\det S \ne 0$, and matrix $U = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ is diagonal with eigenvalues of K^2 on its main diagonal, and $|\lambda_i| = 1$, $i = 1, \dots, n$. Matrix U is defined uniquely up to permutation of diagonal elements and matrix S is defined uniquely up to permutation of rows. Put $Q = S^{-1}VS$, where $V = \operatorname{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots, \sqrt{\lambda_n})$. It is obvious that $Q^2 = K^2$ and Q is orthogonal matrix defined up to values of roots $\sqrt{\lambda_i}$. We claim that there are no other orthogonal matrices whose square equals K^2. Indeed, suppose that there exists an orthogonal matrix P such that $P \ne Q$ and $P^2 = K^2$. Then $P = T^{-1}WT$, where $W = \operatorname{diag}(\mu_1, \mu_2, \dots, \mu_n)$, $|\mu_i| = 1$, $i = 1, \dots, n$. Hence $P^2 = T^{-1} \operatorname{diag}(\mu_1^2, \mu_2^2, \dots, \mu_n^2) T = K^2$. Since representation $K^2 = S^{-1}US$ is unique up to permutation of diagonal elements in U and rows in S, we see that $\{\mu_1^2, \mu_2^2, \dots, \mu_n^2\} = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ and T coincides with S up to permutation of rows. Therefore, $P = Q$. Thus, identity $K^2 = C^2$ implies representations $K = S^{-1}\operatorname{diag}(\gamma_1, \gamma_2, \dots, \gamma_n)S$ and $C = S^{-1}\operatorname{diag}(\delta_1, \delta_2, \dots, \delta_n)S$ such that $\gamma_i^2 = \delta_i^2 = \lambda_i$, $i = 1, \dots, n$. This implies that $KC = CK$. From expressions (2.1) we get $gf(x) = fg(x) + h$ for all $x \in Q$, where $h = Kd + b - (Cb + d)$. As $gf(Q) = fg(Q) = Q$, we obtain $h = 0$ and $gf(x) = fg(x)$ $(x \in Q)$. $\square$

Remark 3. *For example, transformations g and f of form* (2.1) *satisfying identity $gf(x) = fg(x)$ are rotations around the same axis in $\mathbb{R}^3$. Both identities $gf(x) = fg(x)$ and $g^2(x) = f^2(x)$ hold for transformations of rotation around the same axis in $\mathbb{R}^3$ by angles α and $\pi + \alpha$.*

Lemma 6. *Suppose that $G_g^2 \neq \varnothing$, $G_f^2 \neq \varnothing$, $g(Q) = f(Q) = Q$, transformations $g(x)$ and $f(x)$ have form* (2.1), *and identity $fg(x) = gf(x)$ holds for all $x \in Q$. Then operator A is normal.*

Proof. By definition, $D(A) = \{u \in W_2^2(Q) : Bu = 0\}$. Therefore, under conditions of the lemma we get $A_1u, A_2u, A_1^*u, A_2^*u \in D(A)$ if $u \in D(A)$. Then by the theorem on smoothness of generalized solutions of elliptic equations near the boundary ([9], Ch. 2, Sec. 5, Theorem 5.1) we obtain $D(AA^*) = D(A^*A) = \{u \in W_2^4(Q) : Bu = B\Delta u = 0\}$.

By virtue of Lemma 2 it is sufficient to prove that

$$\begin{aligned}(4.31)\quad A_0(A_1 + A_2)u &+ (A_1^* + A_2^*)A_0u \\ &= (A_1 + A_2)A_0u + A_0(A_1^* + A_2^*)u, \quad u \in D(AA^*).\end{aligned}$$

By conditions of the lemma, transformations g and f have form (2.1). In the same way as in the proof of Lemma 5 we get (4.26) and (4.27). These equalities imply (4.31). $\square$

Example 4. *Let us consider an example of operator A being not normal when the condition for commutativity of transformations g and f is not fulfilled while the other conditions of Lemma 6 hold. Consider domain Q and transformations g and f introduced in Example 1. In this case all the conditions of Lemma 6 are fulfilled except for the condition of commutativity. Put $u(x_1, x_2, x_3) = (x_1 + x_2)\xi(x)$, where $\xi \in \dot{C}^\infty(\mathbb{R}^3)$ is a cut-off function such that $0 \le \xi \le 1$, $\xi(x) = 1$ for $x \in Q_{2\varepsilon}$, and $\xi(x) = 0$ for $x \notin Q_\varepsilon$. (Here $Q_\varepsilon \subset Q$ and $\operatorname{dist}(\partial Q_\varepsilon, \partial Q) > \varepsilon$.) It is obvious that $u \in \mathcal{D}(\mathcal{A}\mathcal{A}^*)$ and the left-hand and the right-hand sides of Eq.* (4.31) *become zeros. Then the normality of A becomes equivalent to the normality of $A_1 + A_2$ for this function u. Since all the conditions of Lemma 2 also are fulfilled except for the condition of commutativity, it follows from Example 1 that operator $A_1 + A_2$ is normal if and only if Eq.* (3.2) *holds for all $v \in L_2(Q)$ and $x \in Q$. It is obvious that $u \in L_2(Q)$. Choosing $x = (0,\ 0,\ 1)^T$ and taking into account* *calculations made in Example 1, we see that Eq.* (3.2) *is violated. Therefore, under such conditions operator A is not normal.*

Lemmas 4, 5, and 6 prove Theorem 1.

5. Proof of Theorem 2.

LEMMA 7. *Suppose that $G_g^2 = \varnothing$ and $G_f^2 = \varnothing$. Then $g(Q) = f(Q) = Q$. Let operator A be normal and condition $a_1 + a_2 \neq 0$ hold. Then*:

$$(5.1)\quad \begin{cases} a_1^2\,(|J_g(x)| - |J_g(x)|^{-1}) + a_2^2\,(|J_f(x)| - |J_f(x)|^{-1}) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x \in G_g^1 \cap G_f^1, \\ |J_g(x)| = |J_f(x)| = 1, \quad x \in Q \setminus (G_g^1 \cap G_f^1). \end{cases}$$

Note that the second relation from (5.1) is a special case of the first.

Proof. First, we prove that $G_g^2 = \varnothing$ implies $g(Q) = Q$. Since under the initial assumptions $g(Q) \subset Q$, it is sufficient to prove that $Q \subset g(Q)$. Indeed, for any $x \in Q$ we have $x = g^2(x)$ whence $g^{-1}(x) = g(x) \in Q$. Therefore, any point $x \in Q$ has an original in Q. Thus, $g^{-1}(Q) \subset Q$ whence $Q \subset g(Q)$. Analogously, $G_f^2 = \varnothing$ implies $f(Q) = Q$.

As operator A is normal, we have $D(AA^*) = D(A^*A)$ and Eq. (4.25) holds.

1. Suppose that for some point $x^0 \in G_g^1 \cap G_f^1$ relation $g(x^0) \neq f(x^0)$ holds (and so $g^{-1}(x^0) \neq f^{-1}(x^0)$). Sets G_g^1 and G_f^1 are open, therefore, there exists a neighborhood $B_{2\delta}(x^0)$ such that $\overline{B_{2\delta}(x^0)} \subset G_g^1 \cap G_f^1$. Since transformations g and f are smooth, choose $\delta > 0$ such that $B_{2\delta}(x^0) \cap g(B_{2\delta}(x^0)) = \varnothing$, $B_{2\delta}(x^0) \cap f(B_{2\delta}(x^0)) = \varnothing$, $g(B_{2\delta}(x^0)) \cap f(B_{2\delta}(x^0)) = \varnothing$, and $f^{-1}(B_{2\varepsilon}(x^1)) \cap g^{-1}(B_{2\varepsilon}(x^1)) = \varnothing$. Introduce a cut-off function $\xi \in \dot{C}^\infty(\mathbb{R}^n)$ such that $0 \leq \xi(x) \leq 1$, $\xi(x) = 1$ for $x \in B_\delta(x^0)$, and $\operatorname{supp}\xi \subset B_{2\delta}(x^0)$. It is obvious that $\overline{g^{-1}(B_{2\delta}(x^0))} \subset Q$ and $\overline{f^{-1}(B_{2\delta}(x^0))} \subset Q$. Therefore, $\xi \in D(AA^*)$.

Combining Eq. (4.25) for $u = \xi(x)$, condition $g(x) \neq f(x)$ for $x \in B_{2\delta}(x^0)$, definition of $\xi(x)$, and identities $g(x) = g^{-1}(x)$ and $f(x) = f^{-1}(x)$ for $x \in Q$, we obtain for $x \in B_\delta(x^0)$

$$(5.2)\qquad a_1^2|J_{g^{-1}}(g(x))| + a_2^2|J_{f^{-1}}(f(x))| = a_1^2|J_{g^{-1}}(x)| + a_2^2|J_{f^{-1}}(x)|.$$

By virtue of properties $|J_g(x)| \cdot |J_{g^{-1}}(g(x))| = 1$ and $|J_f(x)| \cdot |J_{f^{-1}}(f(x))| = 1$ for $x \in Q$ and identities $g(x) = g^{-1}(x)$ and $f(x) = f^{-1}(x)$ for $x \in Q$, from Eq. (5.2) we obtain the first of relations (5.1):

$$a_1^2\left(|J_g(x)| - |J_g(x)|^{-1}\right) + a_2^2\left(|J_f(x)| - |J_f(x)|^{-1}\right) = 0, \quad x \in B_\delta(x^0).$$

2. Suppose that for some point $x^0 \in G_g^1 \cap G_f^1$ there exists ε such that $g(x) = f(x)$ for $x \in B_{2\varepsilon}(x^0)$ (and so $g^{-1}(x) = f^{-1}(x)$). Choose $\delta \le \varepsilon$ such that $\overline{B_{2\delta}(x^0)} \subset G_g^1 \cap G_f^1$, $B_{2\delta}(x^0) \cap g(B_{2\delta}(x^0)) = \varnothing$, and $B_{2\delta}(x^0) \cap f(B_{2\delta}(x^0)) = \varnothing$.

We get $g^{-1}(x) = f^{-1}(x)$ for $x \in g(B_{2\delta}(x^0))$. By definition, $g(\widetilde{G}_g^1) = \widetilde{G}_g^1$, therefore, $g(G_g^1) = G_g^1$. Hence $g(B_{2\delta}(x^0)) \subset G_g^1 \cap G_f^1$. Since $G_g^2 = G_f^2 = \varnothing$, we have $g(x) = g^{-1}(x)$ and $f(x) = f^{-1}(x)$ for $x \in Q$. Therefore,

$$g(x) = g^{-1}(x) = f(x) = f^{-1}(x), \qquad x \in B_{2\delta}(x^0) \cup g(B_{2\delta}(x^0)), \tag{5.3}$$

and $[B_{2\delta}(x^0) \cup g(B_{2\delta}(x^0))] \subset G_g^1 \cap G_f^1$.

Introduce a cut-off function $\xi \in C^\infty(\mathbb{R}^n)$ such that $0 \le \xi(x) \le 1$, $\xi(x) = 1$ for $x \in B_\delta(x^0)$, and $\operatorname{supp} \xi \subset B_{2\delta}(x^0)$. Taking into account Eq. (5.3), for $x \in B_\delta(x^0)$ we get:

$$\begin{aligned}
&A_1 A_2^* \eta(x) = a_1 a_2 |J_{f^{-1}}(g(x))| \xi(f^{-1} g(x)) = a_1 a_2 |J_{f^{-1}}(g(x))| = a_1 a_2 |J_g(g(x))|,\\
&A_2 A_1^* \eta(x) = a_1 a_2 |J_{g^{-1}}(f(x))| \xi(g^{-1} f(x)) = a_1 a_2 |J_{g^{-1}}(f(x))| = a_1 a_2 |J_g(g(x))|,\\
&A_1^* A_2 \eta(x) = a_1 a_2 |J_{g^{-1}}(x)| \xi(f g^{-1}(x)) = a_1 a_2 |J_{g^{-1}}(x)| = a_1 a_2 |J_g(x)|,\\
&A_2^* A_1 \eta(x) = a_1 a_2 |J_{f^{-1}}(x)| \xi(g f^{-1}(x)) = a_1 a_2 |J_{f^{-1}}(x)| = a_1 a_2 |J_g(x)|.
\end{aligned}$$

Combining these expressions with Eq. (4.25) for $u = \xi(x)$ and taking into account Eq. (5.3), for $x \in B_\delta(x^0)$ we get

$$\begin{aligned}
(a_1 + a_2)^2 |J_g(g(x))| &= (a_1 + a_2)^2 |J_g(x)|,\\
(a_1 + a_2)^2 |J_f(f(x))| &= (a_1 + a_2)^2 |J_f(x)|.
\end{aligned}$$

Using properties $|J_g(x)| \cdot |J_{g^{-1}}(g(x))| = 1$ and $|J_f(x)| \cdot |J_{f^{-1}}(f(x))| = 1$ for $x \in Q$ and taking into account that $a_1 + a_2 \ne 0$, we get $|J_g(x)| = |J_f(x)| = 1$ for $x \in B_\delta(x^0)$.

From parts **1** and **2** and from the smoothness of $|J_g(x)|$ and $|J_f(x)|$ it follows that the first relation from (5.1) holds for all $x \in G_g^1 \cap G_f^1$.

3. By definition, $f(x) = x$ holds for $x \in \widetilde{G}_f^1$, therefore, $g(x) \ne f(x)$ for $x \in G_g^1 \setminus G_f^1$. Suppose that $G_g^1 \setminus \overline{G_f^1}$ is not empty. As this set is open, we get $|J_f(x)| = 1$ for $x \in G_g^1 \setminus \overline{G_f^1}$. For any point $x^0 \in G_g^1 \setminus \overline{G_f^1}$ there exists a neighborhood $B_{2\delta}(x^0)$, such that $\overline{B_{2\delta}(x^0)} \subset G_g^1 \setminus \overline{G_f^1}$. Choose $\delta > 0$ such that $B_{2\delta}(x^0) \cap g(B_{2\delta}(x^0)) = \varnothing$, and $g(B_{2\delta}(x^0)) \cap f(B_{2\delta}(x^0)) = \varnothing$. Considering equation analogous to Eq. (5.2), we obtain $a_1^2 |J_{g^{-1}}(g(x))| = a_1^2 |J_{g^{-1}}(x)|$ for $x \in B_\delta(x^0)$. Combining this with $|J_g(x)| \cdot |J_{g^{-1}}(g(x))| = 1$ and $g(x) = g^{-1}(x)$, we get $|J_g(x)| = 1$ for $x \in B_\delta(x^0)$. Since point x^0 is arbitrary, this holds for $x \in G_g^1 \setminus \overline{G_f^1}$. In the set $G_f^1 \setminus \overline{G_g^1}$ we obtain the analogous result: $|J_f(x)| = 1$.

By definition, $Q \setminus (G_g^1 \cup G_f^1) = \{x \in Q : g(x) = x, f(x) = x\}$. Therefore, if this set has a non-empty interior, we have $|J_g(x)| = 1$ and $|J_f(x)| = 1$ there.

Sets G_g^1 and G_f^1 are open, therefore, for any point $y \in \partial G_g^1$ or $y \in \partial G_f^1$ at the limit we get $|J_g(y)| = 1$ and $|J_f(y)| = 1$. □

LEMMA 8. *Suppose that* $G_g^2 = \varnothing$ *and* $G_f^2 = \varnothing$. *Let conditions* $|J_g(x)| = 1$ *and* $|J_f(x)| = 1$ *hold for* $x \in Q$. *Then operator* A *is self-adjoint.*

Proof. Under conditions of this lemma it follows from Lemma 1 (taking into account that $g(x) = g^{-1}(x)$ and $f(x) = f^{-1}(x)$ for $x \in Q$) that $A_1 = A_1^*$ and $A_2 = A_2^*$. Since operator A_0 is self-adjoint, operator A also is self-adjoint. □

Lemmas 7 and 8 prove Theorem 2.

6. Proof of Theorem 3.

LEMMA 9. *Suppose that* $G_g^2 \neq \varnothing$, $G_f^2 = \varnothing$, *and* $g(Q) = Q$. *Let operator* A *be normal. Then*:

1). $g(x) = Kx + b$, $x \in Q$, *where* K *is an orthogonal matrix of* $n \times n$ *size*, $K^2 \neq E$, *and* $b \in \mathbb{R}^n$.

2). $f(Q) = Q$ *and* $|J_f(x)| = 1$, $x \in Q$.

Proof. First, we prove the first statement of the lemma. Consider transformation g. Analogously to the proof of Lemma 3, choose a point $x^0 \in G_g^2$. Properties $(A1)$ and $(A2)$ hold at this point. There exists a neighborhood $B_{2\delta}(x^0) \subset G_g^2$ satisfying properties $(B1)$ and $(B2)$.

1. In the same way as in the part **1** of the proof of Lemma 3, suppose that at the point x^0 properties $(A3)$–$(A8)$ hold, therefore, we can choose $\delta > 0$ sufficiently small so that corresponding properties $(B3)$–$(B8)$ hold. Since condition $G_f^2 = \varnothing$ implies that $f(x) = f^{-1}(x)$ for $x \in Q$, properties $(A4)$ and $(A5)$ coincide as well as properties $(B4)$ and $(B5)$ corresponding them.

All the further reasoning from part **1** of the proof of Lemma 3 remains valid. According to it, transformation g is linear in the neighborhood $B_\delta(x^0)$ of the point x^0 and is described by Eq. (4.10).

2. Analogously to the proof of Lemma 3, we consider different cases where properties $(A3)$–$(A8)$ are violated. All the cases where properties $(A3)$, $(A6)$, $(A7)$, and $(A8)$ get violated one at a time are considered in the same way as in the corresponding parts **2**, **5**, **6**, and **7** of the proof of Lemma 3. The case where either property $(A4)$ or $(A5)$ get violated is considered in the same way as in part **10** of the proof of Lemma 3 because these properties are equivalent under conditions of the present lemma. In the last case no conditions on coefficients a_1 and a_2 is used.

Due to relation $f(x) = f^{-1}(x)$ for all $x \in Q$ and the equivalency of $(\overline{A4})$ and $(\overline{A5})$, only the following combinations of properties $(\overline{A3})$–$(\overline{A8})$ are possible: $(\overline{A4})$ and $(\overline{A5})$ (already considered) , $(\overline{A6})$ and $(\overline{A8})$, $(\overline{A6})$ and $(\overline{A7})$, $(\overline{A3})$ and $(\overline{A7})$, and $(\overline{A3})$ and $(\overline{A8})$ (see part **8** of the proof of Lemma 3). All the combinations except for the first are considered in the same way as in part **9** of the proof of Lemma 3 and reduced to combinations of corresponding particular cases.

As a result, the proof of Lemma 3 is repeated taking into account condition $f(x) = f^{-1}(x)$ for $x \in Q$ which proves the first statement of the present lemma.

Now we prove the second statement of the lemma. Consider transformation f. By virtue of condition $G_f^2 = \varnothing$ we get $f(x) = f^{-1}(x)$ for all $x \in Q$ and $f(Q) = Q$ (as was proved in Lemma 7). Choose any point $x^0 \in G_g^1 \cap G_f^1$ and a neighborhood $B_{2\delta}(x^0)$ such that $\overline{B_{2\delta}(x^0)} \subset G_g^1 \cap G_f^1$, $B_{2\delta}(x^0) \cap g(B_{2\delta}(x^0)) = \varnothing$, $B_{2\delta}(x^0) \cap g^{-1}(B_{2\delta}(x^0)) = \varnothing$, and $B_{2\delta}(x^0) \cap f(B_{2\delta}(x^0)) = \varnothing$.

3. Suppose that $f(x^0) \neq g(x^0)$ and $f^{-1}(x^0) \neq g^{-1}(x^0)$. Then choose $\delta > 0$ such that also $g(B_{2\delta}(x^0)) \cap f(B_{2\delta}(x^0)) = \varnothing$ and $f^{-1}(B_{2\varepsilon}(x^1)) \cap g^{-1}(B_{2\varepsilon}(x^1)) = \varnothing$. Introduce a cut-off function $\xi \in \dot{C}^\infty(\mathbb{R}^n)$ such that $0 \leq \xi(x) \leq 1$, $\xi(x) = 1$ for $x \in B_\delta(x^0)$, and $\operatorname{supp} \xi \subset B_{2\delta}(x^0)$. It is obvious that $\xi \in D(A^*A)$.

Since operator A is normal (as expressed by Eq. (4.25)), by definition of $B_{2\delta}(x^0)$ and $u = \xi(x)$ we obtain Eq. (5.2) for $x \in B_\delta(x^0)$. By the first statement of the present lemma, transformation g has form (2.1). Therefore, $|J_g(x)| = |J_{g^{-1}}(x)| = 1$ for $x \in Q$. Taking this into account as well as identities $f(x) = f^{-1}(x)$ and $|J_f(x)| \cdot |J_{f^{-1}}(x)| = 1$ for $x \in Q$, from Eq. (5.2) we get $|J_f(x)| = 1$ for $x \in B_\delta(x^0)$.

4. Suppose that $f(x) = g(x)$ and $f^{-1}(x) \neq g^{-1}(x)$ for $x \in S_1 \subset G_g^1 \cap G_f^1$, where S_1 is a set either with non-empty or empty interior. Then by the first statement of this lemma we get $f(x) = g(x) = Kx + b$ for $x \in S_1$, where K is an orthogonal matrix of $n \times n$ size such that $K^2 \neq E$ and $b \in \mathbb{R}^n$. Therefore, $|J_f(x)| = \det K = 1$ for $x \in S_1 \setminus \partial S_1$.

5. Suppose that $f^{-1}(x) = g^{-1}(x)$ and $f(x) \neq g(x)$ for $x \in S_2 \subset G_g^1 \cap G_f^1$, where S_2 is a set either with non-empty or empty interior. Then we get $f(x) = f^{-1}(x) = g^{-1}(x) = K^{-1}x - K^{-1}b$ for $x \in S_2$. Therefore, $|J_f(x)| = \det K^{-1} = 1$ for $x \in S_2 \setminus \partial S_2$.

6. Suppose that $f(x) = g(x)$ and $f^{-1}(x) = g^{-1}(x)$ for $x \in S_3 \subset G_g^1 \cap G_f^1$, where S_3 is some set. Then $g^2(x) = x$ for $x \in S_3$, i. e., $S_3 \subset \widetilde{G}_g^2$. By the first statement of this lemma, we get $\widetilde{G}_g^2 \subset \partial G_g^2$ (also see part **11** of the proof of Lemma 3). Therefore, $S_3 \subset \partial G_g^2$. Set S_3 has empty interior and we shall consider it below in part **9** of this proof.

7. By definition, for $x \in Q \setminus G_f^1$ we have $f(x) = x$ whence $|J_f(x)| = 1$ for $x \in Q \setminus \overline{G_f^1}$.

8. By definition, for $x \in G_f^1 \setminus G_g^1$ we have $f(x) \neq x$ and $g(x) = x$. By the first statement of the lemma, we get $\widetilde{G}_g^1 = \{z \in Q : z = Kz + b\}$, therefore, set $\widetilde{G}_g^1$ belongs to a hyperplane of dimension $r \leq n-1$, where r is an order of eigenvalue 1 of matrix K. This implies that $\widetilde{G}_g^1 \subset \partial G_g^1$ whence $G_f^1 \setminus G_g^1 \subset \partial G_g^1$, therefore, set $G_f^1 \setminus G_g^1$ has empty interior and will be considered in part **9** of this proof.

9. Parts **3**–**5** and **7** imply that $|J_f(x)| = 1$ for all $x \in Q$ excluding some set with empty interior. This set consists of ∂S_1, ∂S_2, S_3, and ∂G_f^1. By the smoothness of $|J_f(x)|$, at the limit we also get $|J_f(x)| = 1$ in this set. □

LEMMA 10. *Suppose that $G_g^2 \neq \varnothing$, $G_f^2 = \varnothing$, and $g(Q) = Q$. Let operator A be normal. Then $gf(x) = fg(x)$, $x \in Q$.*

Proof. From Lemma 9 it follows that transformations g and f have form (2.3).

By definition, $D(A) = \{u \in W_2^2(Q) : Bu = 0\}$. As was proved in Lemma 7, condition $G_f^2 = \varnothing$ implies $f(Q) = Q$. Then by conditions $g(Q) = f(Q) = Q$ and (2.3) we get $A_1u, A_2u, A_1^*u, A_2^*u \in D(A)$ if $u \in D(A)$. Therefore by the theorem on smoothness of generalized solutions of elliptic equations near the boundary ([9], Chap. 2, Sec. 5, Theorem 5.1) we obtain $D(AA^*) = D(A^*A) = \{u \in W_2^4(Q) : Bu = B\Delta u = 0\}$.

Since transformation g has form (2.3), equations (4.3)–(4.4) hold identically in Q. In the same way as in the proof of Lemma 5, combining this with Eq. (4.2), we get

$$A_0A_1u = A_1A_0u, \qquad A_0A_1^*u = A_1^*A_0u, \qquad u \in D(AA^*). \tag{6.1}$$

On the other hand, under conditions of the lemma we have $G_f^2 = \varnothing$, whence $f(x) = f^{-1}(x)$ for all $x \in Q$. Taking this into account, from (2.3) we get $A_2 = A_2^*$. Therefore,

$$A_0A_2u = A_0A_2^*u, \qquad A_2A_0u = A_2^*A_0u, \qquad u \in D(AA^*). \tag{6.2}$$

Combining equations (6.1) and (6.2) with Eq. (4.25), in the same way as in the proof of Lemma 5 we get Eq. (4.28). Repeating the reasoning from the proof of Lemma 5, we prove that for any $x \in Q$ at least one of equalities $g^2(x) = f^2(x)$ and $gf(x) = fg(x)$ holds. But the first of them can only be fulfilled when $x \in \widetilde{G}_g^2$ because $f^2(x) = x$ for all $x \in Q$. By Lemmas 3 and 9, $\widetilde{G}_g^2 \subset \partial G_g^2$. Therefore, identity $gf(x) = fg(x)$ holds for $x \in G_g^2$ and by the smoothness of g and f it is extended to $\overline{G_f^2} = Q$. □

LEMMA 11. *Suppose that $G_g^2 \neq \varnothing$, $G_f^2 = \varnothing$, and $g(Q) = Q$. Let condition $fg(x) = gf(x)$ hold for $x \in Q$ and let transformations g and f have form*

$$g(x) = Kx + b, \quad x \in Q,$$
$$|J_f(x)| = 1, \quad x \in Q.$$

where K is an orthogonal matrix of $n \times n$ size, $K^2 \neq E$ and $b \in \mathbb{R}^n$. Then operator A is normal and $f(Q) = Q$.

Proof. As was proved in Lemma 7, statement $f(Q) = Q$ follows from condition $G_f^2 = \varnothing$. In the same way as in the proof of Lemma 6, we get $D(AA^*) = D(A^*A) = \{u \in W_2^4(Q) : Bu = B\Delta u = 0\}$. All the conditions of Lemma 2 are fulfilled, therefore, operator $A_1 + A_2$ is normal in $L_2(Q)$. Now it is sufficient to prove that Eq. (4.31) holds.

It can be proved that operator A_2 is self-adjoint in the same way as in the proof of Lemma 8. So $A_2 = A_2^*$ in $L_2(Q)$. In the same way as in the proof of Lemma 6, we obtain Eq. (4.26). This proves Eq. (4.31) and completes the proof. □

Lemmas 9, 10, and 11 prove Theorem 3.

Acknowledgements. The author is grateful to Professor A.L.Skubachevskii for his attention to this work and useful discussions.

REFERENCES

[1] Vorontsov, M. A., Ivanov, V.Yu., and Smalhausen, V.I., Rotary instability in the spatial structure of light fields in nonlinear media with two-dimensional feedback. *Laser Optics in Condensed Matter.* Plenum Press, New York, 1988, 507–517.

[2] Vorontsov M. A., Iroshnikov N. G., and Abernathy R. L., Diffractive patterns in nonlinear optical two-dimensional feedback system with field rotation. *Chaos, Solitons and Fractals.* 1994, **4**, 1701–1716.

[3] Razgulin, A. V., On autooscillations in nonlinear parabolic problem with transformed argument. *U.S.S.R. J. of Comput. Math. and Math. Physics*, 1993, **33**, **1**, 69–80; English trans. *Comp. Math. and Math. Physics*, Pergamon Press, Great Britain, 1993.

[4] Razgulin, A. V., Rotational multi-petal waves in optical system with 2-D feedback. In *Chaos in Optics*, ed. Rajarshi Roy. *Proceedings SPIE*, 1993, **2039**, 342–352.

[5] Skubachevskii, A. L., On some properties of elliptic and parabolic functional differential equations. *Uspekhi Mat. Nauk*, 1996, **51**, 169–170; English trans. *Russian Math. Surveys*, 1996, **51**.

[6] Skubachevskii, A. L., Bifurcation of periodic solutions for nonlinear parabolic functional differential equation arising in optoelectronics. *Nonlinear Anal., Theory Methods Appl.*, Pergamon Press, Great Britain, 1998, **32**, No 2, 261–278.

[7] Skubachevskii, A. L., On the normality of some elliptic functional differential operators. *Funkts. Anal. Prilozh.*, 1997, **31**, No. 4, 60–65; English trans. *Funct. Anal. Appl.*, 1997, **31**.

[8] Skubachevskii, A. L., On the Hopf bifurcation for quasi-linear parabolic functional differential equation. *Differ. Uravn.*, 1998, **34**, No 10, 1394–1401; English trans. *Differ. Equations*, 1998, **34**.

[9] Lions, J.-L. and Magenes, E., *Problemès aux limites non-homogènes et applications*, Vol. 1. Dunod, Paris, 1968; English trans. Springer-Verlag, Berlin, 1972.