The normality of some elliptic functional-differential operators of second order

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In non-linear optical systems with a transformation of the field in two-dimensional feedback, there arise various regular periodic phenomena called "multipetal waves" [1], [2]. These light structures are used in modern computer technology for creating optical analogues of neuron networks. Such systems are modelled mathematically in terms of bifurcations of periodic solutions of quasi-linear parabolic functional-differential equations with a transformation $g(x)$ of the space variables. This problem was considered in [3], [4] in the case when the domain Q is a disc or annulus and g is a rotation through some angle θ. The case when $Q \subset \mathbb{R}^2$ and g are arbitrary was studied in [5], [6]. It was assumed in these papers that the linearized elliptic functional-differential operator is normal. Necessary and sufficient conditions for this operator to be normal were obtained in [7] in terms of $Q \subset \mathbb{R}^n$ and g. More general cases without the assumption of normality were investigated in [8].

Here we obtain necessary and sufficient conditions for the normality of the linearized operator in the case of two transformations of the space variables.

Let $Q \subset \mathbb{R}^n$ be a bounded domain with boundary $\partial Q \subset C^\infty$, $n \geqslant 2$. Let g, f be one-to-one transformations of class C^3 such that

$$g\colon V \subset \mathbb{R}^n \to g(V) \subset \mathbb{R}^n, \qquad |J_g(x)| \neq 0, \quad x \in V,$$
$$f\colon V \subset \mathbb{R}^n \to f(V) \subset \mathbb{R}^n, \qquad |J_f(x)| \neq 0, \quad x \in V.$$

Here V is the bounded domain with $\overline{Q} \subset V$, $J_g(x) = [\partial g_i/\partial x_j]_{i,j=1}^n$ is the Jacobian matrix of g and $|J_g(x)| = |\det J_g(x)|$. Also let $g(Q) \subset Q$ and $f(Q) \subset Q$.

Consider the unbounded operator $A_0\colon L_2(Q) \to L_2(Q)$ with domain of definition

$$\mathscr{D}(A_0) = \{v \in W_2^2(Q) : Bv = 0\}$$

and given by $A_0 v = \Delta v$, $v \in \mathscr{D}(A_0)$. Here $W_2^k(Q)$ is the Sobolev space of complex-valued functions lying in $L_2(Q)$ along with all their generalized derivatives up through order k, and $Bv = v\big|_{\partial Q}$ or $Bv = (\partial v/\partial \nu)\big|_{\partial Q}$, where ν is the unit inner normal vector to ∂Q at the point $x \in \partial Q$. It is known that A_0 is self-adjoint. Put $A\colon L_2(Q) \to L_2(Q)$, $A = A_0 + A_1 + A_2$, where A_1, A_2 are bounded linear operators defined on the whole space $L_2(Q)$ as follows:

$$A_1\colon L_2(Q) \to L_2(Q), \quad A_1 v(x) = a_1 v(g(x)),$$
$$A_2\colon L_2(Q) \to L_2(Q), \quad A_2 v(x) = a_2 v(f(x)),$$

where $a_1 \neq 0$, $a_2 \neq 0$ are real numbers.

An operator A is said to be normal if $\mathscr{D}(AA^*) = \mathscr{D}(A^*A)$ and $AA^*v = A^*Av$ for all $v \in \mathscr{D}(A^*A)$. We put $D(A) = D(A_0)$.

Consider the sets $G_g^m = \{x \in Q : g^m(x) \neq x\}$, $m = 1, 2, \dots$, where $g^m(x)$ denotes the result of applying g m times. Write $\widetilde{G}_g^m = Q \setminus G_g^m$. We also write the superposition of transformations in the form $fg(x)$, $g^{-1}f(x)$ and so on.

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Our main results are as follows.

Theorem 1. *Assume that* $G_g^2 \neq \varnothing$, $G_f^2 \neq \varnothing$, $g(Q) = f(Q) = Q$ *and* $|a_1| \neq |a_2|$. *Then* A *is normal if and only if*

$$g(x) = Kx + b, \quad f(x) = Cx + d, \quad x \in Q,$$
$$gf(x) = fg(x), \quad x \in Q,$$

where K, C *are orthogonal matrices of order* n, $K^2 \neq E$, $C^2 \neq E$ *and* $b, d \in \mathbb{R}^n$.

Theorem 2. *Assume that* $G_g^2 = \varnothing$, $G_f^2 = \varnothing$. *Then* $g(Q) = f(Q) = Q$ *and*:

1) *if* A *is normal and* $a_1 + a_2 \neq 0$,

$$\begin{cases} a_1^2\big(|J_g(x)| - |J_g(x)|^{-1}\big) + a_2^2\big(|J_f(x)| - |J_f(x)|^{-1}\big) = 0, & x \in G_g^1 \cap G_f^1, \\ |J_g(x)| = |J_f(x)| = 1, & x \in Q \setminus (G_g^1 \cap G_f^1); \end{cases}$$

2) *if* $|J_g(x)| = |J_f(x)| = 1$ *for* $x \in Q$, A *is a normal self-adjoint operator.*

Theorem 3. *Assume that* $G_g^2 \neq \varnothing$, $G_f^2 = \varnothing$ *and* $g(Q) = Q$. *Then* $f(Q) = Q$, *and* A *is normal if and only if*

$$g(x) = Kx + b, \quad |J_f(x)| = 1, \quad x \in Q,$$
$$gf(x) = fg(x), \quad x \in Q,$$

where K *is an orthogonal matrix of order* n, $K^2 \neq E$ *and* $b \in \mathbb{R}^n$.

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