Andronov–Hopf bifurcation for quasi-linear parabolic functional differential equations with transformations of spatial variables

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A non-linear optical system with field transformations in a two-dimensional feedback loop is described by the following Neumann problem [1]:

$$u_t + u = D\Delta u + K\biggl(1 + \sum_{i=1}^{N} \gamma_i \cos u_{g_i}\biggr), \qquad x \in Q, \quad t \in \mathbb{R}, \tag{1}$$

$$(\partial u/\partial \widetilde{\nu})\big|_{\partial Q \times \mathbb{R}} = 0. \tag{2}$$

Here $Q \subset \mathbb{R}^n$ is a bounded domain with boundary $\partial Q \in C^\infty$; $D, K, \gamma_1, \dots, \gamma_N \in \mathbb{R}$ are non-zero constant coefficients; $u_{g_i} = u(g_i(x), t)$, $g_i\colon \overline{Q} \to g_i(\overline{Q})$ are one-to-one transformations, $i = 1, \dots, N$; ν is the outward unit vector normal to ∂Q at the point x and $\widetilde{\nu} = (\nu, 0)$.

For applications it is important to investigate Andronov–Hopf bifurcation of periodic solutions of the equation (1). In many papers (for example, [2], [3]) this problem has been considered in the case when the domain Q is a disc or annulus and the only transformation of variables is a rotation by a constant angle and a contraction. The case of an arbitrary domain Q and a single transformation of variables was investigated in [4]. In [5] the normality of the linearized operator of the problem was used. The normality of this operator was studied in [6] for the problem with two transformations of variables.

In this paper we generalize the results of [4] to the case of any finite number of transformations of variables. We use the approach developed in [7], [8] for investigating Andronov–Hopf bifurcation in infinite-dimensional problems.

Condition 1. $g_i(Q) \subseteq Q$, $g_i(x) \not\equiv x$ $(x \in Q)$, $i = 1, \dots, N$.

Condition 2. *The operators* $G_i\colon L_p(Q) \to L_p(Q)$ *with* $(G_i u)(x) = u(g_i(x))$ *are bounded*, $i = 1, \dots, N$.

A solution w of the problem (1), (2) is called a *spatio-homogeneous stationary solution* if it does not depend on $x \in Q$ nor $t \in \mathbb{R}$. It satisfies the transcendental equation

$$w = K\biggl(1 + \sum_{i=1}^{N} \gamma_i \cos w\biggr). \tag{3}$$

Condition 3. $1 + \widehat{K} \sin \widehat{w} \sum_{i=0}^{N} \gamma_i \neq 0$, *where* $\widehat{w}$ *is the solution of* (3) *for* $K = \widehat{K}$.

Let X be a real Banach space. We denote by $C^\sigma_{2\pi}(X)$, $0 < \sigma < 1$, the space of all 2π-periodic functions $\varphi\colon \mathbb{R} \to X$ that are σ-continuous in the Hölder sense, equipped with the norm

$$\|\varphi\|_{C^\sigma_{2\pi}(X)} = \sup_{0 \leqslant t \leqslant 2\pi} \|\varphi(t)\|_X + \sup_{0 \leqslant s < t \leqslant 2\pi} \|\varphi(t) - \varphi(s)\|_X/(t-s)^\sigma.$$

Let $C^{1,\sigma}_{2\pi}(X)$ be the Banach space of all differentiable functions $\varphi\colon \mathbb{R} \to X$ such that φ and φ' belong to $C^\sigma_{2\pi}(X)$, with the norm $\|\varphi\|_{C^{1,\sigma}_{2\pi}(X)} = \sup_{0 \leqslant t \leqslant 2\pi} \|\varphi(t)\|_X + \|\varphi'\|_{C^\sigma_{2\pi}(X)}$.

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Let $W_p^k(Q)$ ($\widetilde{W}_p^k(Q)$) be the Sobolev space of real-valued (complex-valued) functions belonging to $L_p(Q)$ ($\widetilde{L}_p(Q)$) together with their generalized derivatives up to order k. Let $W_{p,N}^2(Q) = \{v \in W_p^2(Q) : (\partial v/\partial \nu)|_{\partial Q} = 0\}$ and $\widetilde{W}_{p,N}^2(Q) = \{v \in \widetilde{W}_p^2(Q) : (\partial v/\partial \nu)|_{\partial Q} = 0\}$.

Suppose that $w = w(\varkappa)$ satisfies (3) for $K = \widehat{K} + \varkappa$ and $w(0) = \widehat{w}$. We represent a solution of the problem (1), (2) as $u(x,t,\varkappa) = w(\varkappa) + v(x,t,\varkappa)$. Then (1) takes the form

$$v_t = f(v, \varkappa),$$

where $f(v,\varkappa) = D\Delta v - v + (\widehat{K} + \varkappa)\sum_{i=1}^N \gamma_i\bigl(\cos(w(\varkappa) + v_{g_i}) - \cos w(\varkappa)\bigr)$.

We define the operator $\mathscr{L}(\varkappa) : \mathscr{D}(\mathscr{L}(\varkappa)) \subset L_p(Q) \to L_p(Q)$ with domain $\mathscr{D}(\mathscr{L}(\varkappa)) = W_{p,N}^2(Q)$ by the formula $\mathscr{L}(\varkappa)v = f_v(0,\varkappa)v = D\Delta v - v - (\widehat{K} + \varkappa)\sin w(\varkappa)\sum_{i=1}^N \gamma_i v_{g_i}$. Let us consider the operator $\widetilde{\mathscr{L}}(\varkappa) : \mathscr{D}(\widetilde{\mathscr{L}}(\varkappa)) \subset \widetilde{L}_p(Q) \to \widetilde{L}_p(Q)$ with domain $\mathscr{D}(\widetilde{\mathscr{L}}(\varkappa)) = \widetilde{W}_{p,N}^2(Q)$ defined by the formula $\widetilde{\mathscr{L}}(\varkappa) = \mathscr{L}(\varkappa)v_1 + i\mathscr{L}(\varkappa)v_2$, where $v_1, v_2 \in \mathscr{D}(\mathscr{L}(\varkappa))$ and $v = v_1 + iv_2$.

Let $\lambda_s(\varkappa) = \delta_s(\varkappa) + i\omega_s(\varkappa)$, $s = 1, 2, \dots$, denote the eigenvalues of the operator $\widetilde{\mathscr{L}}(\varkappa)$.

Condition 4. *$\lambda_1(0) = i\widehat{\omega}$ is a simple eigenvalue of the operator $\widetilde{\mathscr{L}}(0)\colon \mathscr{D}(\widetilde{\mathscr{L}}(\varkappa)) \subset \widetilde{L}_p(Q) \to \widetilde{L}_p(Q)$, $\widehat{\omega} > 0$, $k\widehat{\omega}i \notin \sigma(\widetilde{\mathscr{L}}(0))$ for $k = 0, 2, 3, \dots$, and $\delta_1'(0) \neq 0$.*

We let $\tau = \widehat{\omega}\omega(\varkappa)t$, where $\omega(\varkappa)$ is an unknown frequency close to 1, and we consider the 2π-periodic solutions of the equation

$$v_\tau = (\widehat{\omega}\omega(\varkappa))^{-1} f(v(\tau), \varkappa), \qquad \tau \in \mathbb{R}.$$

The following theorem is the main result of this paper.

Theorem 1. *Assume the conditions 1–4 and let the numbers $\sigma \in (0,1)$ and $p > n/2$ be fixed. Then there exist a number $\varepsilon_0 > 0$, an analytic vector-valued function $\varepsilon \mapsto (v(\varepsilon), \omega(\varepsilon), \varkappa(\varepsilon))$ acting from $(-\varepsilon_0, \varepsilon_0)$ to $C_{2\pi}^\sigma(W_{p,N}^2(Q)) \cap C_{2\pi}^{1,\sigma}(L_p(Q)) \times \mathbb{R} \times \mathbb{R}$ such that $v(0) = 0$, $\omega(0) = 1$, $\varkappa(0) = 0$, and $v(\varepsilon)$ is not constant with respect to τ for $\varepsilon \neq 0$, and an analytic function $\varepsilon \mapsto w(\varkappa(\varepsilon))$ from $(-\varepsilon_0, \varepsilon_0)$ to $\mathbb{R}$.*

The function $u(x,t,\varepsilon) = w(\varkappa(\varepsilon)) + v(x,\tau,\varepsilon)$ is $2\pi(\widehat{\omega}\omega(\varepsilon))^{-1}$-periodic with respect to t and is a solution of the problem (1), (2), where $\tau = \omega(\varepsilon)\widehat{\omega}t$. In addition, $\omega(\varepsilon) = 1 + \varepsilon^2\omega_2 + \varepsilon^3\omega_3 + \cdots$ and $\varkappa(\varepsilon) = \varepsilon^2\varkappa_2 + \varepsilon^3\varkappa_3 + \cdots$.

Moreover, there exists a $\delta_0 > 0$ such that if $\overline{\varkappa}, \overline{\omega} \in \mathbb{R}$ and $\overline{v} \in C_{2\pi}^\sigma(W_{p,N}^2(Q)) \cap C_{2\pi}^{1,\sigma}(L_p(Q))$ satisfy the conditions

$$\overline{v}'(\tau) = (\widehat{\omega}\overline{\omega})^{-1} f(\overline{v}(\tau), \overline{\varkappa}), \quad \tau \in \mathbb{R},$$
$$\|\overline{v}\|_{C_{2\pi}^\sigma(W_{p,N}^2(Q)) \cap C_{2\pi}^{1,\sigma}(L_p(Q))} < \delta_0, \quad |\overline{\varkappa}| < \delta_0, \quad |1 - \overline{\omega}| < \delta_0,$$

then there exist a $\theta \in [0, 2\pi)$ and an $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ such that $\overline{\varkappa} = \varkappa(\varepsilon)$, $\overline{\omega} = \omega(\varepsilon)$, and $\overline{v}(\tau) = v(x, \tau + \theta, \varepsilon)$.

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Bibliography

[1] М. А. Воронцов, Ю. Д. Думаревский, Д. В. Пруидзе, В. И. Шмальгаузен, *Изв. АН СССР. Сер. физ.* **52**:2 (1988), 374–376; English transl., M. A. Vorontsov, Yu. D. Dumarevskii, D. V. Pruidze, and V. I. Shmal'gauzen, *Izv. Acad. Sci. USSR Atmospher. Ocean. Phys.* **52** (1988).

[2] Е. П. Белан, О. Б. Лыкова, *Дифференц. уравнения* **40**:10 (2004), 1348–1357; English transl., E. P. Belan and O. B. Lykova, *Differ. Equ.* **40**:10 (2004), 1419–1430.

[3] А. В. Разгулин, *ЖВМ и МФ* **33**:1 (1993), 69–80; English transl., A. V. Razgulin, *Comput. Math. Math. Phys.* **33**:1 (1993), 61–70.

[4] А. Л. Скубачевский, *Дифференц. уравнения* **34**:10 (1998), 1394–1401; English transl., A. L. Skubachevskii, *Differential Equations* **34**:10 (1998), 1395–1402.

[5] A. L. Skubachevskii, *Nonlinear Anal.* **32**:2 (1998), 261–278.

[6] Е. М. Варфоломеев, *УМН* **61**:1 (2006), 173–174; English transl., E. M. Varfolomeev, *Russian Math. Surveys* **61**:1 (2006), 184–185.

[7] M. G. Crandall and P. H. Rabinowitz, *Arch. Rational Mech. Anal.* **67**:1 (1977), 53–72.

[8] G. Da Prato and A. Lunardi, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **3**:4 (1986), 315–329.

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